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Cover image: A box for your cylinder collection. When a cube is balanced on one corner, all of its corners lie on four evenly spaced horizontal planes; see page 331. The arrangement of the cylinders is from the note by Jerzy Kocik on page 384.

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MATHEMATICS MAGAZINE

LETTER FROM THE EDITOR

The articles in this issue deal with long-standing questions in combinatorics and group theory.

In the first article, Nicholas Pippenger starts with a well-known problem about voltages on the vertices of a cube, and provides, at last, a complete solution. Along the way he illustrates a full set of combinatorial methods, including generating functions, asymptotic expansions, and direct counting arguments. He makes a connection as well to preferential arrangements, which will attract the attention of voting theorists.

Some of us know Cayley graphs mainly from their frequent occurrence as examples in graph theory. In the article by Richard Goldstone, John McCabe, and Kathryn Weld, they become the center of attention in their own right. Does the Cayley Graph determine the group? Not always, it turns out, and in answering the question, the authors draw on the work of some founders of group theory.

The Note by Art Benjamin, Bob Chen, and Kimberly Kindred gives a new proof of an old formula for sums of binomial coefficients. They give a combinatorial argument, but is it really a counting argument? They are counting with complex numbers, not just integers.

Also in the Notes, both Jacob Siehler and Eugene Gover look at sequences that are monotone for a while, and then change direction—but Siehler's sequences are random, while Gover's are definitely not. Christopher Swanson finds deep mathematics in a shouting game, and Robert Lamphere shows us how circular Newtonian orbits would operate in hyperbolic space. Finally, Jerzy Kocik analyzes a pattern of tangent circles, which inspires the image on this month's cover.

This issue marks the end of my first year as Editor. It has been a joy. The best part of the job is reading the many excellent submissions, published and unpublished. We always wish we had more pages. All of the MAA journals depend absolutely on the talent and efforts of all of our authors.

The MAGAZINE also depends on the "thoughtful and measured reflection" of its referees. We are glad to recognize them at page 400. (The quoted phrase is from the Reviews column, which includes some thoughts on referees' roles; see page 398.)

It has been a pleasure to work with the Associate Editors, with the MAA Publications Staff, and with Frank Farris, my predecessor as Editor and now Chair of the MAA's Publications Council. I'm pleased to thank both Swarthmore College and Bryn Mawr College for library access as well as welcoming me into their rich mathematical community. Finally, we appreciate all those who turn our strings of characters into a delivered journal after the editors' work is done.

Walter Stromquist, Editor

ARTICLES

The Hypercube of Resistors, Asymptotic Expansions, and Preferential Arrangements

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A classic puzzle asks for the effective resistance between vertices at the ends of a long diagonal when the edges of a cube are replaced by 1-ohm resistors. The solution relies on the observation that for each of the endpoints, the three adjacent vertices are at the same potential, by the symmetry of the cube under a 120° rotation about the long diagonal. The network is thus equivalent to one in which three resistors in parallel are in series with six resistors in parallel and with three resistors in parallel, for a total effective resistance of $1/3 + 1/6 + 1/3 = 5/6$ ohms. This problem seems first to have appeared in 1914 in a book by Brooks and Poyser [**4**]. It has also appeared in this magazine as Quickie Q32, submitted by Nathan Eisen [**9**], with an alternative solution by C. W. Trigg [**23**]. In case either the problem or its solution seems mysterious, the Appendix offers a rapid review of the relevant part of circuit theory and the use of symmetry to solve its problems.

A natural question is: What happens when the 3-dimensional cube is replaced by an n-dimensional hypercube? This n-dimensional version of the problem was posed in 1976 by Mullin and Zave as Problem E 2620 in the *American Mathematical Monthly* [**14**], with a solution by Jagers [**12**], and again in 1979 by Singmaster as Problem 79-16 in *SIAM Review* [**17**], with a solution by Rennie [**16**]. In this paper, we shall use this more general problem to explore a number of topics involving asymptotic expansion and combinatorial enumeration. We shall conclude by solving the still more general problem (proposed by Singmaster [**17**], but apparently still unsolved) of determining the resistance between vertices at distance k in the n-dimensional hypercube.

First, we consider the resistance between the endpoints of a long diagonal in the n-dimensional hypercube. Reasoning as before, we observe that all the vertices at a given distance from one of the endpoints of the long diagonal are again at the same potential, so the network is equivalent (in the sense that the potential difference across, and current through, each resistor remains unchanged) to a series connection of parallel connections of resistors. Since there are $n + 1$ distances $(0, 1, \ldots, n)$ from one endpoint, there are n parallel connections. There are $\binom{n}{k}$ vertices at distance k from the endpoint, and for $0 \le k \le n - 1$, each of these vertices is connected by $n - k$ resistors to vertices at distance $k + 1$. Therefore the total effective resistance is

$$R_n = \sum_{0 \le k \le n-1} \frac{1}{\binom{n}{k}(n-k)},$$

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and using the identity $\binom{n}{k}(n-k) = n\binom{n-1}{k}$ (which is easily seen using the expressions for binomial coefficients in terms of factorials) we see that this is equivalent to

$$R_n = \frac{1}{n} \sum_{0\le k\le n-1} \frac{1}{\binom{n-1}{k}}. \tag{1}$$

The numbers being summed in (1) are elements of the "harmonic triangle", considered by Leibniz as a companion to the "arithmetic triangle" of Pascal (see Boyer [**3**, p. 439]). In the arithmetic triangle

$$\begin{array}{llllll} 1 \\ 1 & 1 \\ 1 & 2 & 1 \\ 1 & 3 & 3 & 1 \\ 1 & 4 & 6 & 4 & 1 \\ 1 & 5 & 10 & 10 & 5 & 1 \\ \vdots & & & & & \ddots, \end{array}$$

each entry (except the first and last in each row) is the sum of the elements to its north and its north-west, while in the harmonic triangle

$$\begin{array}{llllll} 1/1 \\ 1/2 & 1/2 \\ 1/3 & 1/6 & 1/3 \\ 1/4 & 1/12 & 1/12 & 1/4 \\ 1/5 & 1/20 & 1/30 & 1/20 & 1/5 \\ 1/6 & 1/30 & 1/60 & 1/60 & 1/30 & 1/6 \\ \vdots & & & & & \ddots, \end{array}$$

each entry is the sum of the elements to its south and south-east. The resistance R_n is the sum of the entries in the n-th row:

n :	0	1	2	3	4	5	6	7	8	$\cdots$
R_n :	0	1	1	5/6	2/3	8/15	13/30	151/420	32/105	$\cdots$.

The appearance of the reciprocals of binomial coefficients in (1) suggests that we also consider the sum

$$S_n = \sum_{0\le k\le n} \frac{1}{\binom{n}{k}}, \tag{2}$$

which has the following values:

n :	0	1	2	3	4	5	6	7	8	$\cdots$
S_n :	1	2	5/2	8/3	8/3	13/5	151/60	256/105	83/35	$\cdots$.

Of course, these two sequences are linked by the relations

$$R_n = \frac{1}{n} S_{n-1} \quad \text{and} \quad S_n = (n+1) R_{n+1}. \tag{3}$$

In the next section, we shall review some exact results (alternative expressions and generating functions) involving the numbers R_n and S_n. We shall also consider asymptotic expansions for these numbers; the coefficients in these asymptotic expansions have simple combinatorial interpretations that will launch us on a tour of old and new results in combinatorial enumeration. Finally, we shall return to the hypercube of resistors, and consider the resistance between vertices that are not the endpoints of a long diagonal.

Alternative expressions and generating functions

The numbers R_n and S_n have alternative expressions,

$$R_n = \frac{1}{2^n} \sum_{1 \le k \le n} \frac{2^k}{k} \tag{4}$$

and

$$S_n = \frac{n+1}{2^n} \sum_{0 \le k \le n} \frac{2^k}{k+1}, \tag{5}$$

which are equivalent to each other by virtue of (3). While having just as many terms as (1) and (2), these sums have simpler summands, and will thus lend themselves more easily to further developments.

The first proof of (5) was given by Staver [**19**], who derived from (2) the recurrence $S_n = \big((n+1)/2n\big) S_{n-1} + 1$, from which (5) follows by induction from the base case $S_0 = 1$. The formula (4) was given without proof by Jagers [**12**], and was given with an "electrical" proof by Rennie [**16**], as follows. First, let a current of 1 ampere flow into a vertex A and out of a vertex B long-diagonally opposite to A. If B is at potential 0, then A is at potential R_n volts. Let A' be adjacent to A, and B' long-diagonally opposite to A', and therefore adjacent to B. By symmetry, $1/n$ amperes flows through the 1-ohm resistor from A to A', so A' is at potential $R_n - 1/n$ volts. By a similar argument, B' is at potential $1/n$ volts. Second, reconnect the current source so that 1 ampere flows into A' and out of B'. If B' is at potential 0, then A' is at potential R_n volts, A is at potential $R_n - 1/n$ volts, and B is at potential $1/n$ volts. Third, suppose that currents of 1 ampere flow in at each of A and A', and out of each of B and B'. By linearity, we may superimpose the potentials and subtract $1/n$ from their sum, putting B and B' at potential 0, and A and A' at potential $2R_n - 2/n$ volts. But with this final current distribution, 1 ampere flows from A to B' through the resistors of an $(n-1)$-dimensional hypercube, 1 ampere flows from A' to B through the resistors of another disjoint $(n-1)$-dimensional hypercube, and no current flows through the resistors connecting corresponding vertices in the two hypercubes, since by symmetry they are at equal potentials. Thus if B and B' are at potential 0, A and A' are at potential R_{n-1} volts. We therefore have $2R_n - 2/n = R_{n-1}$, or $R_n = (1/2)R_{n-1} + 1/n$, from which (4) follows by induction from the base case $R_0 = 0$. Finally, we mention that Sury [**21**] proved (5) by using the integral representation $\int_0^1 x^k (1-x)^{n-1-k}\, dx = 1/\left(n\binom{n-1}{k}\right)$ (Euler's beta integral), summing the resulting geometric progression inside the integral, and evaluating the resulting integral by a change of variable.

Equations (4) and (5) allow us easily to derive the generating functions $R(z) = \sum_{n \ge 0} R_n\, z^n$ and $S(z) = \sum_{n \ge 0} S_n\, z^n$ for the sequences R_n and S_n. Indeed, since $-\log(1-z) = z + z^2/2 + z^3/3 + \cdots + z^k/k + \cdots$, we see that $2^k/k$ is the coefficient of z^k in $-\log(1-2z)$ (when $k \ge 1$). If $A(z) = \sum_{n \ge 0} A_n\, z^n$ and $B(z) = \sum_{n \ge 0} B_n\, z^n$ are the generating functions for the sequences A_n and B_n, respectively, then $C(z) = A(z)\, B(z)$ is the generating function for the sequence $C_n = \sum_{0 \le k \le n} A_k\, B_{n-k}$, called the "convolution" of the sequences A_n and B_n. As a special case, $B(z) = 1/(1-z)$ is the generating function for the sequence $B_n = 1$, so that $A(z)/(1-z)$ is the generating function for the sequence $\sum_{0 \le k \le n} A_k$ of partial sums of the sequence A_n. Thus

$\sum_{1\le k\le n} 2^k/k$ is the coefficient of z^n in $\big(-\log(1-2z)\big)/(1-z)$, and $2^{-n}\sum_{1\le k\le n} 2^k/k$ is the coefficient of z^n in $\big(-\log(1-z)\big)/(1-\frac{1}{2}z)$ (even for $k=0$), so that

$$R(z) = \frac{1}{1-\frac{1}{2}z}\log\frac{1}{1-z}. \tag{6}$$

From (3), we see that $S(z) = R'(z)$, so differentiating (6) yields

$$S(z) = \frac{1}{(1-z)(1-\frac{1}{2}z)} + \frac{1}{2(1-\frac{1}{2}z)^2}\log\frac{1}{1-z}. \tag{7}$$

Generating functions for sums similar to (4) and (5) have been given by Pla [**15**].

Asymptotic expansions

The results of the preceding section give exact values of R_n and S_n as rational numbers but they yield little insight into the behavior of these sequences for large n. To obtain this insight, we develop asymptotic expansions. It will be convenient to use "O-notation", where $O\big(f(n)\big)$ stands for some function $g(n)$ (possibly a different function at each occurrence) such that $|g(n)| \le c\, f(n)$ for some constant c and all sufficiently large n.

We start with (1). Since the binomial coefficients $\binom{n-1}{k}$ increase as k increases from 0 to $\lfloor (n-1)/2 \rfloor$, then decrease as k increases from $\lceil (n-1)/2 \rceil$ to $n-1$, the largest terms in (1) are the first and last: $1/\binom{n-1}{0} = 1/\binom{n-1}{n-1} = 1$. The next largest terms are the second and second-to-last, which are $1/\binom{n-1}{1} = 1/\binom{n-1}{n-2} = O(1/n)$. There are $n-4$ other terms, and each of these is at most $1/\binom{n-1}{2} = 1/\binom{n-1}{n-3} = O(1/n^2)$, so the sum of all these other terms is also $O(1/n)$. Thus we have

$$R_n = \frac{2}{n} + O\left(\frac{1}{n^2}\right). \tag{8}$$

This result gives a good estimate of R_n when n is large.

We can refine the estimate (8) by including the second and second-to-last terms exactly. Now the third and third-to-last terms are $O(1/n^2)$, and each of the remaining $n-6$ terms is $O(1/n^3)$, so their sum is also $O(1/n^2)$. This yields

$$R_n = \frac{2}{n}\left(1 + \frac{1}{n-1} + O\left(\frac{1}{n^2}\right)\right).$$

Continuing in this way, we obtain

$$R_n = \frac{2}{n}\left(1 + \frac{1}{(n-1)} + \frac{2}{(n-1)(n-2)} + \cdots + \frac{k!}{(n-1)(n-2)\cdots(n-k)} + O\left(\frac{1}{n^{k+1}}\right)\right) \tag{9}$$

for any fixed k. This derivation is valid when $n \ge 2k$ (otherwise, we are double-counting terms).

Equation (9) gives a sort of asymptotic expansion for R_n, but its content would be clearer if the denominator of each term were a power of n, instead of the "falling powers" $(n-1)(n-2)\cdots(n-k)$ that appear there. That is, we would like an expansion of the form

$$R_n = \frac{2}{n}\left(r_0 + \frac{r_1}{n} + \frac{r_2}{n^2} + \cdots + \frac{r_k}{n^k} + O\left(\frac{1}{n^{k+1}}\right)\right) \tag{10}$$

for each $k \geq 0$. It is customary to write

$$\frac{n\,R_n}{2} \sim r_0 + \frac{r_1}{n} + \frac{r_2}{n^2} + \cdots + \frac{r_k}{n^k} + \cdots \tag{11}$$

as shorthand for the assertion of (10) for each $k \geq 0$. The series (11) is called an *asymptotic expansion*; it is not convergent for any n (since the $k!$ in the numerators grows faster than the exponential n^k in the denominator), but it allows R_n to be approximated with an error $O(1/n^k)$ for any fixed k and all sufficiently large n (where the constant hidden in the O-notation depends on k).

Our task is to determine the coefficients $r_0, r_1, \ldots$ in (11). To do this we shall expand each term $k!/(n-1)(n-2)\cdots(n-k)$ in (9) into a series of negative powers of n,

$$\frac{k!}{(n-1)(n-2)\cdots(n-k)} = \sum_{\ell \geq k} \frac{k!\,t_{k,\ell}}{n^\ell}, \tag{12}$$

then sum the contributions to r_ℓ for each $k \leq \ell$. First, we consider the numbers $t_{k,\ell}$ in the expansion (12). They are what have come to be called the "Stirling numbers of the second kind", for which we shall use the notation suggested by Knuth [**13**, p. 65]: $t_{k,\ell} = \left\{ {\ell \atop k} \right\}$. These numbers were introduced by James Stirling in the introduction to his *Methodus Differentialis* [**20**] in 1730. He defined them as the numbers that express a power z^ℓ of z as a linear combination of the polynomials $z, z(z-1), \ldots, z(z-1)\cdots(z-\ell+1)$:

$$z^\ell = \sum_{0 \leq k \leq \ell} \left\{ {\ell \atop k} \right\} z(z-1)\cdots(z-k+1),$$

and he gave a table for $1 \leq k \leq \ell \leq 9$. The number $\left\{ {\ell \atop k} \right\}$ has a simple combinatorial interpretation: it is the number of ways to partition the ℓ elements of the set $L = \{1, \ldots, \ell\}$ into k blocks (non-empty subsets of ℓ that are pairwise disjoint and whose union is ℓ). For $\ell = 3$, for example, we have one partition $\{\{1, 2, 3\}\}$ into one block, three partitions $\{\{1\}, \{2, 3\}\}$, $\{\{1, 2\}, \{3\}\}$ and $\{\{1, 3\}, \{2\}\}$ into two blocks and one partition $\{\{1\}, \{2\}, \{3\}\}$ into three blocks; thus $\left\{ {3 \atop 1} \right\} = 1$, $\left\{ {3 \atop 2} \right\} = 3$ and $\left\{ {3 \atop 3} \right\} = 1$. At the end of his introduction, Stirling gives the expansion

$$\frac{1}{(z+1)(z+2)\cdots(z+k)} = \sum_{\ell \geq k} (-1)^{\ell-k} \left\{ {\ell \atop k} \right\} \frac{1}{z^\ell},$$

which, upon substitution of $-n$ for z, gives (12) in the form

$$\frac{k!}{(n-1)(n-2)\cdots(n-k)} = \sum_{\ell \geq k} k! \left\{ {\ell \atop k} \right\} \frac{1}{n^\ell}.$$

These expansions are in fact convergent for fixed $k \geq 1$ and for $|z| > 1$, or fixed $k \geq 1$ and $n > k$, though Stirling did not distinguish convergent expansions, such as these, and asymptotic expansions, such as (11). Applying this result to each term in (9) gives the desired asymptotic expansion:

$$\frac{n\,R_n}{2} \sim \sum_{\ell \geq 0} \left(\sum_{0 \leq k \leq \ell} \left\{ {\ell \atop k} \right\} k! \right) \frac{1}{n^\ell}.$$

Thus the coefficients r_ℓ we sought are given by

$$r_\ell = \sum_{0 \le k \le \ell} \left\{ \begin{matrix} \ell \\ k \end{matrix} \right\} k!. \tag{13}$$

It will doubtless have occurred to the reader that if there are "Stirling numbers of the second kind", there should also be "Stirling numbers of the first kind". Indeed there are, and they were also introduced by Stirling [**20**]. He defined them as the numbers that express $z(z+1)\cdots(z+\ell-1)$ as a linear combination of the polynomials $z, z^2, \ldots, z^\ell$. Nowadays it is more common to define them as the absolute values of the numbers that expand $z(z-1)\cdots(z-\ell+1)$ as a linear combination of the polynomials $z, z^2, \ldots, z^\ell$; in the notation of Knuth [**13**, p. 65]:

$$z(z-1)\cdots(z-\ell+1) = \sum_{0 \le k \le \ell} (-1)^{\ell-k} \left[\begin{matrix} \ell \\ k \end{matrix} \right] z^k.$$

Stirling again gave a table for $1 \le k \le \ell \le 9$, and the expansion

$$\frac{1}{z^k} = \sum_{\ell \ge k} \left[\begin{matrix} \ell \\ k \end{matrix} \right] \frac{1}{z(z+1)\cdots(z+k-1)}.$$

These numbers too have a simple combinatorial interpretation: $\left[\begin{smallmatrix} \ell \\ k \end{smallmatrix} \right]$ is the number of permutations of ℓ elements that have k cycles. For $\ell = 3$, for example, we have two permutations (123) and (132) with one cycle, three permutations (1)(23), (12)(3) and (13)(2) with two cycles and one permutation (1)(2)(3) with three cycles; thus $\left[\begin{smallmatrix} 3 \\ 1 \end{smallmatrix} \right] = 2$, $\left[\begin{smallmatrix} 3 \\ 2 \end{smallmatrix} \right] = 3$ and $\left[\begin{smallmatrix} 3 \\ 3 \end{smallmatrix} \right] = 1$. (Because of their combinatorial interpretations, the Stirling numbers of the first and second kinds are sometimes called the Stirling cycle numbers and Stirling partition numbers, respectively.)

We can find a similar asymptotic expansion for S_n. Again noting that the largest terms in the sum (2) are the first and the last, we obtain

$$S_n = 2 + O\left(\frac{1}{n}\right).$$

Generalizing this result as before yields

$$S_n = 2\left(1 + \frac{1}{n} + \frac{2}{n(n-1)} + \cdots + \frac{k!}{n(n-1)\cdots(n-k+1)} + O\left(\frac{1}{n^{k+1}}\right)\right).$$

Applying (12) to each term and summing the contributions for each negative power of n, we obtain

$$\frac{S_n}{2} \sim s_0 + \frac{s_1}{n} + \frac{s_2}{n^2} + \cdots + \frac{s_k}{n^k} + \cdots, \tag{14}$$

where

$$s_\ell = \sum_{0 \le k \le \ell} \left\{ \begin{matrix} \ell \\ k \end{matrix} \right\} (k+1)!.$$

The coefficients r_ℓ and s_ℓ have simple combinatorial interpretations that we shall study in the following section.

Preferential arrangements

In this section we shall study the numbers r_ℓ and s_ℓ. Our model for this study will be a collection of results concerning the "exponential numbers" d_ℓ, given by

$$d_\ell = \sum_{0\le k\le \ell} \left\{ {\ell \atop k} \right\}. \tag{15}$$

These numbers have a simple combinatorial interpretation: d_ℓ is the number of ways to partition the set $\{1, \ldots, \ell\}$ into any number of blocks. For $\ell = 3$, for example, we have seen that there is one partition into one block, three partitions into two blocks and one partition into three blocks; thus $d_3 = 1 + 3 + 1 = 5$. We have the table

ℓ :	0	1	2	3	4	5	6	7	8	$\cdots$
d_ℓ :	1	1	2	5	15	52	203	877	4140	$\cdots$.

(The sequence d_ℓ is A000110 in Sloane's *On-Line Handbook of Integer Sequences* [**18**].)

There are three aspects of the exponential numbers that are of particular interest to us: a recurrence, a generating function and an expression as the sum of an infinite series. The recurrence is

$$d_\ell = \delta_\ell + \sum_{0\le k\le \ell-1} \binom{\ell-1}{k} d_k, \tag{16}$$

where δ_ℓ is 1 for $\ell = 0$, and 0 for all other values of ℓ. This recurrence allows d_ℓ to be computed from the previous values $d_0, d_1, \ldots, d_{\ell-1}$. (We could avoid the use of δ_ℓ by stating the initial condition $d_0 = 1$ separately, but it will be convenient to have recurrences that hold for all values of ℓ.)

The "exponential generating function" $d(z) = \sum_{\ell\ge 0} d_\ell\, z^\ell/\ell!$ is given by

$$d(z) = e^{e^z-1}, \tag{17}$$

where the term "exponential" refers to the factor $1/\ell!$ in the defining sum (in contrast to the "ordinary" generating functions that we used for the numbers R_n in (6) and S_n in (7)).

The expression as an infinite sum is

$$d_\ell = \frac{1}{e} \sum_{n\ge 0} \frac{n^\ell}{n!}, \tag{18}$$

where $e = 2.7182\ldots$ is the base of natural logarithms. Since d_ℓ is expressed as a finite sum in (15), it may not be clear what advantage there is to (18). But the finite sum involves the Stirling numbers of the second kind, whereas the infinite sum involves only powers and factorials. Furthermore, the sum (18) has a simple probabilistic interpretation: d_ℓ is the ℓ-th moment (that is, the expectation $\mathrm{Ex}[N^\ell]$ of the ℓ-th power N^ℓ) of a Poisson-distributed random variable N with mean $\lambda = 1$ (since $\Pr[N = n] = e^{-\lambda}\,\lambda^n/n! = e^{-1}(1/n!)$ for such a random variable).

The exponential numbers were mentioned in 1934 by Bell [**1**] [**2**], and are on that account sometimes called the "Bell numbers". But (16), (17) and (18) were all given earlier: (16) in 1933 by Touchard [**22**] (in an equivalent "umbral" form), (17) in 1886 by Whitworth [**24**, Proposition XXIV, p. 95] and (18) in 1877 by Dobiński [**8**] (who actually only gave the cases $1 \le \ell \le 8$; but it is clear from his derivations that d_ℓ satisfies the recurrence (16)).

In his marvelous book *Asymptotic Methods in Analysis*, N. G. de Bruijn [**5**, Section 3.3] derives the asymptotic expansion

$$\frac{1}{n!}\sum_{0\le k\le n} k! \sim 1+\frac{d_0}{n^1}+\frac{d_1}{n^2}+\cdots+\frac{d_k}{n^{k+1}}+\cdots,$$

and then says that it is "only for the sake of curiosity" that he mentions that the coefficients d_ℓ, given by (15), have a combinatorial interpretation. One of our goals in this paper is to pursue this curiosity; our motto is: whenever the coefficients in an expansion are integers, look for a combinatorial interpretation!

The coefficients r_ℓ, given by (13), also have a simple combinatorial interpretation: they are the number of ways of ranking ℓ candidates, with ties allowed; that is, the ℓ candidates are first to be partitioned into equivalence classes, then the equivalence classes are to be linearly ordered. This interpretation follows from those of $\left\{ {\ell \atop k} \right\}$ and $k!$, where k is the number of equivalence classes in the partition. Because of this interpretation, r_ℓ is called the number of *preferential arrangements* of ℓ elements. For $\ell = 3$, for example, the one partition into one block can have its block ordered in one way, each of the three partitions into two blocks can have its blocks ordered in two ways and the partition into three blocks can have its blocks ordered in six ways; thus $r_3 = 1\cdot 1 + 3\cdot 2 + 1\cdot 6 = 13$. We have the table

ℓ :	0	1	2	3	4	5	6	7	8	$\cdots$
r_ℓ :	1	1	3	13	75	541	4683	47293	545835	$\cdots$.

(The sequence r_ℓ is A000670 in Sloane [**18**].) We shall derive the recurrence

$$r_\ell = \delta_\ell + \sum_{0\le k\le \ell-1}\binom{\ell}{k} r_k, \tag{19}$$

the exponential generating function (defined by $r(z) = \sum_{\ell\ge 0} r_\ell\, z^\ell/\ell!$)

$$r(z) = \frac{1}{2-e^z} \tag{20}$$

and the summation expression

$$r_\ell = \frac{1}{2}\sum_{n\ge 0}\frac{n^\ell}{2^n}. \tag{21}$$

We begin by deriving the recurrence (19). For $\ell \ge 1$, we can construct a preferential arrangement on ℓ candidates by first choosing the number k of candidates tied in the top equivalence class (with k in the range $1 \le k \le \ell$), then choosing in one of $\binom{\ell}{k}$ ways the candidates in this class, and finally choosing in one of $r_{\ell-k}$ ways a preferential arrangement of the remaining $\ell - k$ candidates. This gives us

$$r_\ell = \sum_{1\le k\le \ell}\binom{\ell}{k} r_{\ell-k}.$$

Making the substitution $k = j + 1$, then the substitution $j = \ell - 1 - k$, and finally using the identity $\binom{\ell}{\ell-k} = \binom{\ell}{k}$, we obtain

$$\begin{aligned} r_\ell &= \sum_{1 \le k \le \ell} \binom{\ell}{k} r_{\ell-k} \\ &= \sum_{0 \le j \le \ell-1} \binom{\ell}{j+1} r_{\ell-j-1} \\ &= \sum_{0 \le k \le \ell-1} \binom{\ell}{\ell-k} r_k, \\ &= \sum_{0 \le k \le \ell-1} \binom{\ell}{k} r_k. \end{aligned}$$

This equation holds for $\ell \ge 1$; since $r_0 = 1$, we obtain (19) for $\ell \ge 0$.

Next we shall derive the exponential generating function (20). Adding r_ℓ to both sides of (19) yields

$$2r_\ell = \delta_\ell + \sum_{0 \le j \le \ell} \binom{\ell}{j} r_j. \tag{22}$$

Multiplying both sides of this equation by $z^\ell/\ell!$ and summing over $\ell \ge 0$, then making the substitution $k = \ell - j$ and using the identity $e^z = \sum_{k \ge 0} z^k/k!$, we obtain

$$\begin{aligned} 2r(z) &= 1 + \sum_{\ell \ge 0} \frac{z^\ell}{\ell!} \sum_{0 \le j \le \ell} \binom{\ell}{j} r_j \\ &= 1 + \sum_{j \ge 0} \frac{z^j\, r_j}{j!} \sum_{k \ge 0} \frac{z^k}{k!} \\ &= 1 + e^z\, r(z). \end{aligned} \tag{23}$$

Solving this equation for $r(z)$ yields (20).

Finally we shall derive the summation expression (21). To do this, we rewrite the exponential generating function $r(z)$ from (20):

$$\begin{aligned} \sum_{\ell \ge 0} \frac{r_\ell\, z^\ell}{\ell!} &= \frac{1}{2} \frac{1}{1 - \frac{1}{2} e^z} \\ &= \frac{1}{2} \sum_{n \ge 0} \frac{e^{nz}}{2^n} \\ &= \frac{1}{2} \sum_{n \ge 0} \frac{1}{2^n} \sum_{\ell \ge 0} \frac{(nz)^\ell}{\ell!} \\ &= \sum_{\ell \ge 0} \left(\frac{1}{2} \sum_{n \ge 0} \frac{n^\ell}{2^n} \right) \frac{z^\ell}{\ell!}. \end{aligned}$$

Since the coefficient of $z^\ell/\ell!$ must be the same on both sides of this equation, we obtain (21).

The name "preferential arrangement" was introduced by Gross [**10**], as was the summation expression (21). The numbers r_ℓ (with a different combinatorial interpretation involving trees), the recurrence (19) and the generating function (20) were given by Cayley [**6**] in 1859; the combinatorial interpretation we have used is implicit in 1866 by Whitworth [**24**, Proposition XXII, p. 93] (Whitworth shows that the terms $\left\{ {\ell \atop k} \right\} k!$ for fixed k have the exponential generating function $(e^z - 1)^k$; summation over $k \ge 0$ then yields $1/\bigl(1 - (e^z - 1)\bigr) = 1/(2 - e^z)$.)

We turn now to the numbers s_ℓ, which also have a simple combinatorial interpretation: s_ℓ is the number of ways of ranking ℓ candidates, with ties allowed, and with a "bar" that may be placed above all the candidates, between two equivalence classes of tied candidates, or below all the candidates. Thus we may call s_ℓ the number of *barred preferential arrangements* of ℓ elements. If there are k equivalence classes of tied candidates, there are $k + 1$ positions for the bar. For $l = 3$, for example, the one preferential arrangement with one block has two positions for the bar, each of the six preferential arrangements with two blocks has three positions for the bar, and each of the six preferential arrangements with three blocks has four positions for the bar; thus $s_3 = 1 \cdot 2 + 6 \cdot 3 + 6 \cdot 4 = 44$. We have the table

ℓ :	0	1	2	3	4	5	6	7	8	$\cdots$
s_ℓ :	1	2	8	44	308	2612	25988	296564	3816548	$\cdots$.

(The sequence s_ℓ is A005649 in Sloane [**18**].)

Instead of a recurrence for the numbers s_n, we shall derive a formula expressing them in terms of the numbers r_ℓ:

$$s_\ell = \sum_{0 \le k \le \ell} \binom{\ell}{k} r_k \, r_{\ell-k}. \tag{24}$$

We shall also derive the exponential generating function (which is defined by $s(z) = \sum_{\ell \ge 0} s_\ell \, z^\ell / \ell!$)

$$s(z) = \frac{1}{(2 - e^z)^2} \tag{25}$$

and the summation expression

$$s_\ell = \frac{1}{4} \sum_{n \ge 0} \frac{(n+1) n^\ell}{2^n}. \tag{26}$$

For $\ell \ge 0$, we can construct a barred preferential arrangement on ℓ candidates by first choosing the number k of candidates above the bar (with k in the range $0 \le k \le l$), then choosing in one of $\binom{\ell}{k}$ ways the candidates above the bar, then choosing in one of r_k ways a preferential arrangement of these candidates, and finally choosing in one of $r_{\ell-k}$ ways a preferential arrangement of the remaining $\ell - k$ candidates. This gives the formula (24). Next, multiplying both sides of (24) by $z^\ell/\ell!$ and summing over $\ell \ge 0$ yields

$$\begin{aligned} s(z) &= \sum_{\ell \ge 0} \frac{z^\ell}{\ell!} \sum_{0 \le k \le \ell} \binom{\ell}{k} r_k \, r_{\ell-k} \\ &= \sum_{k \ge 0} \frac{z^k \, r_k}{k!} \sum_{j \ge 0} \frac{z^j \, r_j}{j!} \\ &= r(z)^2, \end{aligned} \tag{27}$$

where we have made the substitution $k = \ell - j$. Substituting (20) in this equation yields (25). Finally, reasoning similar to that used to derive (21) leads to (26).

Before concluding this section, let us derive two more identities relating r_ℓ and s_ℓ:

$$r_{\ell+1} = \sum_{0 \le k \le \ell} \binom{\ell}{k} s_k \tag{28}$$

and

$$r_\ell + r_{\ell+1} = 2s_\ell. \tag{29}$$

These can be given direct combinatorial proofs (and the reader may enjoy finding these), but we shall use two different methods that are often useful when dealing with sequences that have explicit exponential generating functions.

To prove (28), we use the notion of "binomial convolution". Suppose that $a(z) = \sum_{\ell \ge 0} a_\ell \, z^\ell/\ell!$ and $b(z) = \sum_{\ell \ge 0} b_\ell \, z^\ell/\ell!$ are the exponential generating functions for the sequences a_ℓ and b_ℓ, respectively. Then

$$\begin{aligned} a(z)\, b(z) &= \sum_{k \ge 0} \frac{a_k \, z^k}{k!} \sum_{j \ge 0} \frac{b_j \, z^j}{j!} \\ &= \sum_{\ell \ge 0} \frac{z^\ell}{\ell!} \sum_{0 \le k \le \ell} \binom{\ell}{k} a_k \, b_{\ell-k}, \end{aligned}$$

where we have made the substitution $j = \ell - k$. Thus $c(z) = a(z)\, b(z)$ is the exponential generating function for the sequence $c_\ell = \sum_{0 \le j \le \ell} \binom{\ell}{j} a_j \, b_{\ell-j}$, which is called the *binomial convolution* of the sequences a_ℓ and b_ℓ, and denoted $(a * b)_\ell$. (We have already encountered binomial convolutions twice in this section: once to derive (23) from (22), where the convolution can be expressed as $s = \delta + r * \upsilon$, where υ is the sequence defined by $\upsilon_\ell = 1$ for all $\ell \ge 0$, which has exponential generating function $\upsilon(z) = e^z$; and again to derive (27) from (24).) We shall also need the fact that $a'(z)$ (where the prime indicates differentiation) is the exponential generating function for the sequence $a_{\ell+1}$, which we shall denote a'_ℓ.

To derive (28), we may now observe that $r'(z) = e^z/(2 - e^z)^2 = e^z \, s(z)$. Thus $r_{\ell+1} = r'_\ell = (s * \upsilon)_\ell$, which yields (28).

To derive (29), we note that $r'(z) = e^z/(2 - e^z)^2$ implies that $r(z)$ satisfies the differential equation

$$r'(z) + r(z) = 2r(z)^2. \tag{30}$$

(This differential equation, together with the initial condition $r(0) = 1$, uniquely determines $r(z)$. In fact, it is an example of a "Bernoulli equation", which can be solved analytically for $r(z)$; see Ince [**11**, p. 22].) Substituting (27) in (30), we obtain (29). (We note that (29) can also be obtained from (21) and (26).)

That the numbers s_ℓ, defined by (14), have the exponential generating function given in (25) was given as an exercise (without proof or reference) by Comtet [**7**, p. 294, Ex. 15]. Our combinatorial interpretation of these numbers in terms of barred preferential arrangements seems to be new.

More resistances

We mentioned in the introduction that Singmaster [**17**] posed in 1978 the problem of determining R_n. In fact, Singmaster asked not only for R_n, but for $R_{n,k}$, the resistance between two vertices at distance k (for $1 \le k \le n$) in an n-dimensional hypercube of 1-ohm resistors. Rennie's solution [**16**] to Singmaster's problem covered (by various arguments) the cases $k = 1, 2$ and $k = n, n-1, n-2$. We shall finish this paper by giving (for the first time, and by a single argument) a complete solution to Singmaster's problem: for $0 \le k \le n$,

$$R_{n,k} = \frac{2}{n} \sum_{0 \le j \le k-1} \frac{1}{\binom{n-1}{j}} \frac{1}{2^n} \sum_{j+1 \le i \le n} \binom{n}{i}. \tag{31}$$

Our solution, like that of Rennie, is based on the principles of symmetry and superposition. We shall also show that this solution possesses some attractive mathematical properties: it is an *increasing* function of k (that is, the further apart the vertices are, the greater the resistance between them), and it is a *concave* function of k (that is, the further apart the vertices are, the less the resistance increases with further increase in the distance).

Consider the situation in which a current of 1 ampere flows out of a vertex A, while currents of $1/(2^n - 1)$ amperes flow into each of the $2^n - 1$ other vertices. Symmetry ensures that all $\binom{n}{j}$ vertices at distance j from A are at the same potential. Call this potential U_j volts, where $U_0 = 0$. There are $1/(n\binom{n-1}{j})$ 1-ohm resistors connecting vertices at potential U_j to vertices at potential U_{j+1}, and a total current of $\sum_{j+1 \le i \le n} \binom{n}{i}/(2^n - 1)$ amperes flows through them. By Ohm's law,

$$U_{j+1} - U_j = \frac{1}{n\binom{n-1}{j}} \frac{1}{2^n - 1} \sum_{j+1 \le i \le n} \binom{n}{i},$$

and thus

$$U_k = \frac{1}{n} \sum_{0 \le j \le k-1} \frac{1}{\binom{n-1}{j}} \frac{1}{2^n - 1} \sum_{j+1 \le i \le n} \binom{n}{i},$$

Now let B be a vertex at distance k from A, and consider the situation in which a current of 1 ampere flows into B and currents of $1/(2^n - 1)$ amperes flow out of each of the $2^n - 1$ other vertices. In this situation there is again a potential difference of U_k volts between A and B. By superposition, if a current of $1 + 1/(2^n - 1)$ amperes flows into B and out of A, there will be a potential difference of $2U_k$ between these vertices. Again using Ohm's law, we have

$$R_{n,k} = \frac{2}{1 + 1/(2^n - 1)} \frac{1}{n} \sum_{0 \le j \le k-1} \frac{1}{\binom{n-1}{j}} \frac{1}{2^n - 1} \sum_{j+1 \le i \le n} \binom{n}{i},$$

which yields (31).

It is not obvious that (31) agrees with (1) for $k = n$. We can verify this agreement as follows. Using the identities $\binom{n-1}{j} = \binom{n-1}{n-1-j}$ and $\binom{n}{i} = \binom{n}{n-i}$, we obtain

$$\begin{aligned} R_{n,n} &= \frac{2}{n} \sum_{0 \le j \le n-1} \frac{1}{\binom{n-1}{j}} \frac{1}{2^n} \sum_{j+1 \le i \le n} \binom{n}{i} \\ &= \frac{1}{n} \sum_{0 \le j \le n-1} \frac{1}{\binom{n-1}{j}} \frac{1}{2^n} \sum_{j+1 \le i \le n} \binom{n}{i} + \frac{1}{n} \sum_{0 \le j \le n-1} \frac{1}{\binom{n-1}{j}} \frac{1}{2^n} \sum_{j+1 \le i \le n} \binom{n}{i} \\ &= \frac{1}{n} \sum_{0 \le j \le n-1} \frac{1}{\binom{n-1}{j}} \frac{1}{2^n} \sum_{j+1 \le i \le n} \binom{n}{i} + \frac{1}{n} \sum_{0 \le j \le n-1} \frac{1}{\binom{n-1}{j}} \frac{1}{2^n} \sum_{0 \le i \le j} \binom{n}{i} \end{aligned}$$

$$= \frac{1}{n} \sum_{0 \le j \le n-1} \frac{1}{\binom{n-1}{j}} \frac{1}{2^n} \sum_{0 \le i \le n} \binom{n}{i}$$

$$= \frac{1}{n} \sum_{0 \le j \le n-1} \frac{1}{\binom{n-1}{j}}$$

$$= R_n.$$

Next, let us show that $R_{n,k}$ is an increasing function of k (that is, that $R_{n,k} - R_{n,k-1} > 0$ for $1 \le k \le n$). From (31) we have

$$R_{n,k} - R_{n,k-1} = \frac{1}{n2^{n-1}} \frac{1}{\binom{n-1}{k-1}} \sum_{k \le i \le n} \binom{n}{i}, \tag{32}$$

and the expression on the right-hand side is obviously positive.

Finally, we shall show that $R_{n,k}$ is a concave function of k (that is, that $R_{n,k} - 2R_{n,k-1} + R_{n,k-2} < 0$ for $2 \le k \le n$). From (32) we have

$$R_{n,k} - 2R_{n,k-1} + R_{n,k-2}$$
$$= \frac{1}{n2^{n-1}} \left(\frac{1}{\binom{n-1}{k-1}} \sum_{k \le i \le n} \binom{n}{i} - \frac{1}{\binom{n-1}{k-2}} \sum_{k-1 \le i \le n} \binom{n}{i} \right).$$

The expression on the right-hand side is obviously negative for $k \le n/2 + 1$, since in this case we have $\binom{n-1}{k-2} \le \binom{n-1}{k-1}$ and $\sum_{k \le i \le n} \binom{n}{i} < \sum_{k-1 \le i \le n} \binom{n}{i}$. For the case $k > n/2 + 1$, we factor $\binom{n-1}{k-2}$ out of the expression in parentheses, then move all but the first term from the second sum into the first sum:

$$R_{n,k} - 2R_{n,k-1} + R_{n,k-2}$$
$$= \frac{1}{n2^{n-1}} \frac{1}{\binom{n-1}{k-2}} \left(\frac{k-1}{n-k+1} \sum_{k \le i \le n} \binom{n}{i} - \sum_{k-1 \le i \le n} \binom{n}{i} \right)$$
$$= \frac{1}{n2^{n-1}} \frac{1}{\binom{n-1}{k-2}} \left(\frac{2k-n-2}{n-k+1} \sum_{k \le i \le n} \binom{n}{i} - \binom{n}{k-1} \right).$$

Thus what remains to be proved is that the expression in parenthesis is negative; that is, that

$$\frac{2k-n-2}{n-k+1} \sum_{k \le i \le n} \binom{n}{i} < \binom{n}{k-1}$$

for $k > n/2 + 1$, or equivalently, by the substitution $k = n - j$ and the identity $\binom{n}{i} = \binom{n}{n-i}$,

$$\frac{n-2j-2}{j+1} \sum_{0 \le i \le j} \binom{n}{i} < \binom{n}{j+1} \tag{33}$$

for $0 \le j < n/2 - 1$.

To prove (33), we observe that $0 \le i < n/2 - 1$ implies that

$$\binom{n}{i} \le \frac{i+1}{n-i} \binom{n}{i+1}.$$

Since $(i+1)/(n-i)$ is an increasing function of i, we have

$$\binom{n}{i} \le \left(\frac{j+1}{n-j}\right)^{j-i+1} \binom{n}{j+1}$$

for $i \le j < n/2 - 1$. Thus we may bound the sum in (33) by the sum of a geometric series,

$$\begin{aligned} \sum_{0\le i\le j} \binom{n}{i} &\le \sum_{0\le i\le j} \left(\frac{j+1}{n-j}\right)^{j-i+1} \binom{n}{j+1} \\ &\le \sum_{m\ge 1} \left(\frac{j+1}{n-j}\right)^{m} \binom{n}{j+1} \\ &= \frac{j+1}{n-2j-1} \binom{n}{j+1}. \end{aligned}$$

This inequality proves (33), and thus completes the proof that $R_{n,k}$ is concave.

Appendix

Our goal in this appendix is to present enough classical circuit theory to allow the reader to understand the problems and proofs in this paper. All the problems dealt with in this paper concern networks of interconnected resistors. Such a network can be regarded as a connected undirected graph (which may have multiple edges, but which does not have self-loops) in which a strictly positive *resistance* is associated with each edge.

A fluid called *charge* (measured in units of "coulombs") flows through the edges, driven by the difference in pressure (also called *potential*, measured in units of "volts") between the ends of each edge. The relationship between the potential difference across an edge and the *current* (rate of flow) through the edge (measured in units of coulombs/second, also called "amperes") is given by *Ohm's law*: the potential difference across an edge is equal to the current through the edge times the resistance of the edge (measured in units of volts/ampere, also called "ohms"). Ohm's law determines the local relationship between current and potential difference at each edge. The global relationship is determined by Kirchhoff's laws. *Kirchhoff's voltage law* says that if we consider any two paths between two given vertices, the sums of the potential differences across the edges along each path are equal (or equivalently, the sum of the potential differences across the edges around any cycle is zero). This law allows us to assign potentials to the vertices of the graph by arbitrarily assigning a potential to one vertex, then using the sums of potential differences along paths to assign a potential to each other vertex. (The connectedness of the graph ensure that this procedure assigns a potential to each vertex, and Kirchhoff's voltage law ensures that the result is independent of the paths chosen. The potential differences assigned to vertices are thus determined up to an arbitrary additive constant.) *Kirchhoff's current law* says that the sum of the currents flowing into each vertex (with negative signs for the currents flowing out) is zero. This is consistent with our understanding of current as measuring the rate of flow of a fluid.

Consider the problem of assigning potentials and currents to a graph (in which resistances are already specified) in a way that satisfies all of these laws. All of the constraints specified by Ohm's and Kirchhoff's laws are linear, and an obvious solution is that all currents and potential differences vanish. This is in fact the unique solution:

if the potentials assigned to the vertices were not all equal, we could find a vertex with the minimum potential connected by an edge to a vertex with strictly larger potential. By Ohm's law, a strictly positive current would flow through this edge into the vertex, but no current could flow out of it, contradicting Kirchhoff's current law.

To obtain non-vanishing currents and potential differences, we consider *current excitations* (or simply excitations), whereby Kirchhoff's current law may violated at each vertex by specifying a current flow from an external source into each vertex (with negative signs for currents flowing out), subject to the condition that the sum over all vertices of these external currents is zero. Suppose the graph contains n vertices. With such an excitation, there will be $n-1$ currents (since the specification of the external current into any $n-1$ vertices determines that into the last) determining $n-1$ vertex potentials (since the potential of one vertex can be chosen arbitrarily). The potentials are determined from these currents by linear equations, and the solution will be unique (since the solution to the homogeneous version of the problem, considered in the previous paragraph, is unique).

Given two vertices in a network, we consider the excitation for which a current of 1 ampere flows into one of the vertices and a current of 1 ampere flows out of the other. We define the *effective resistance* of the network between these vertices to be the potential difference between the two vertices with this excitation. We observe that for a graph containing just two vertices, with one edge between them, the effective resistance between these vertices is equal to the resistance of the edge. Thus, if we think of the network as being encased in a "black box", so that we only have access to it through these two vertices, the network behaves in the same way as a single edge between the two vertices, with the resistance of this edge being the effective resistance of the network between the given vertices.

There are two special cases that allow the calculation of effective resistance to be simplified. One is when networks are connected "in series"; that is when each edge in a chain of edges is replaced by a network. In this case, the effective resistance between the endpoints of the chain is the sum of the effective resistances of the networks between their points of attachment. The other is when several networks are connected "in parallel"; that is, when each edge in a multiple edge between two vertices is replaced by a network. In this case, the reciprocal of the effective resistance between the endpoints of the multiple edge is the sum of the reciprocals of the effective resistances of the networks.

Since the relationship between the currents in an excitation and the resulting potential differences is linear, we may use the principle of superposition: if a network is subjected to a sum of several excitations, the resulting potential differences and currents are the sums of those resulting from the individual excitations.

Finally, we may often exploit the symmetries of a network to facilitate the calculation of resistances. Suppose there is a one-to-one mapping of the vertices and edges of the graph that preserves the incidence relation between vertices and edges and the assignment of resistances to edges. Suppose further that network is subjected to an excitation that is also invariant under this mapping. Then the potentials assigned to corresponding vertices must be equal, and thus currents through and potential differences across each edge will be unchanged if these corresponding vertices are identified into a single vertex.

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Summary Using a classic puzzle concerning a cube of resistors as a point of departure, we use its generalization to a hypercube of resistors as an excuse to survey a number of results concerning generating functions, asymptotic expansions, and combinatorial enumeration. We conclude by giving, apparently for the first time, the complete solution to the problem of the hypercube of resistors.

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Ambiguous Groups and Cayley Graphs— A Problem in Distinguishing Opposites

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The term *inverse problem* names a class of mathematical problems in which one attempts to recover complete information from partial information. A Cayley graph is an example of partial information about a group, typically displayed visually. Can we recover, from that partial information, the multiplication table for the group? The astute reader, instinctively visualizing the Cayley diagrams which are frequently introduced in an introductory course in group theory, may wish to leap ahead with an answer of "Yes!". But in this paper we are concerned with Cayley *graphs*, as opposed to Cayley diagrams which are in fact *directed graphs* (called, for short, Cayley *digraphs*), and Cayley graphs, generally, convey less information than do Cayley digraphs.

Cayley diagrams and graphs are named for Arthur Cayley (1821–1895), the British barrister-turned-mathematician who is credited (by Kleiner in this MAGAZINE [**11**]) with being the first to give the abstract definition of a finite group. In a paper published in 1854, [**4**], Cayley defined a finite group as a set, $G = \{1 = g_1, g_2, \ldots, g_n\}$, together with a binary operation, closed on G, satisfying the group axioms:

- Closure: For all $a, b \in G$, the product ab belongs to G.
- Associativity: For all $a, b, c \in G$, $a(bc) = (ab)c$.
- Right and left multiplication by an element in G are both permutations of G: For each $a \in G$ the set of right multiples $Ga = \{ga \mid g \in G\}$ is G itself, and similarly $aG = G$.

The first two axioms are familiar to anyone who has studied group theory. The reader will note that Cayley assumed that the set G came equipped with a two-sided identity $1 \in G$, and that our current formulation of the axiom requiring that every element of G have an inverse in G is a consequence of Axiom 3. Consequently, Cayley's definition is equivalent to the usual one.

The Cayley diagrams we are familiar with are actually digraphs with labeled edges, invented by Cayley to aid in visualizing a group's structure. Starting with a group G, Cayley began by assigning each element of G to a vertex. Next, he selected a set S of generators for G and drew a directed edge between g and h whenever there was some generator $s \in S$ such that $h = gs$. These edges were labeled or colored to indicate the association with the particular generator s. These edge-colored digraphs were known to determine the table for the group operation, as long as the vertex representing the group identity was identified. The references direct readers to additional treatments of Cayley digraphs, both simple [**3, 7**] and detailed [**9**].

In this paper, we visualize groups using *un*directed graphs, based on a similar principle. Historically, these graphs have been called *Cayley graphs* in Cayley's honor.

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The term *Cayley graph* first began to appear in the literature in the early 1960s, and Sabidussi was one of the first, if not the first, to use the term [**13**]. Cayley graphs interest graph theorists because they have lots of symmetry, and they occur frequently in the literature on applications which involve computer networks or expander graphs. These applications, however, will not be our focus.

To construct a Cayley graph, we modify Cayley's construction. Following him, we take the set of elements of the group as the vertex set. Cayley used a directed edge from g to h to mean that you can get from g to h by right multiplication using $s \in S$. To define an undirected graph we need every connection to go back the other way, and for that we need $s^{-1} \in S$. From now on we require that S be *inverse-closed*, meaning that it contain the inverse of all of its elements. On the other hand, unlike Cayley, we will not require that S generate the entire group. We also want to exclude the possibility of edges from a vertex to itself, so S must not contain the identity element of G. We call any nonempty subset S of G a *Cayley set*, provided S is inverse-closed and $1 \notin S$.

DEFINITION. Let G be a group and let S be a Cayley set of G. Let $\text{Cay}(G, S)$ be the graph with vertex set $V_G = G$. Whenever $g, h \in G$ and there exists an element $s \in S$ such that $h = gs$, vertices g and h are connected with an edge, which we denote by $g \sim h$. The graph $\text{Cay}(G, S)$ is a *Cayley graph* for G.

The edges in the Cayley graph for a group G are not directed edges, and it may happen that the Cayley set S is not a generating set. In general, a Cayley graph for a group G contains less information than does the Cayley digraph. How much has been lost? Is there an undirected analogue of the Cayley digraph that determines the multiplication table for the group? Answering this question is the goal of our paper.

When we no longer require that S be a generating set for G the number of possible Cayley graphs for G increases, and a particular Cayley graph might not be connected. In fact, for any group, there are as many Cayley graphs as there are Cayley subsets. FIGURE 1 gives two different Cayley graphs for $\mathbb{Z}_{12}$, the group of integers under addition modulo 12. The reader should check that the sets $\{4, 8\}$ and $\{2, 10\}$ are inverse-closed, and that the associated Cayley graphs are not isomorphic.

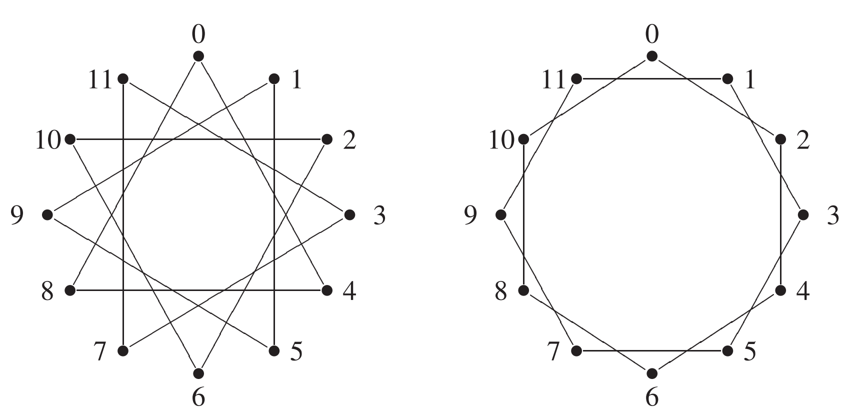

Figure 1 $\text{Cay}(\mathbb{Z}_{12}, \{4, 8\})$ and $\text{Cay}(\mathbb{Z}_{12}, \{2, 10\})$.

For *any* group G of order n, the set $S = G - \{1\}$ is a Cayley set, and the associated Cayley graph $\text{Cay}(G, S)$ is graph-isomorphic to K_n, the complete graph on n vertices, in which each vertex is adjacent to every other vertex. Any two groups of the same order may be represented by this particular Cayley graph, so when we use K_n as a Cayley graph to represent our group, almost all the information about the group has been lost. What if we colored the edges of K_n to indicate their association with a particular Cayley set? The principle is simple: every nontrivial element $g \in G$ is an element of some unique smallest Cayley set C_g. That set will be either a singleton or a doubleton set: if $g = g^{-1}$, then $C_g = \{g\}$, otherwise $g \neq g^{-1}$, and then $C_g = \{g, g^{-1}\}$ contains exactly two elements.

Observe that the collection $\mathcal{A} = \{C_g \mid g \in G,\ g \neq 1\}$ forms a partition of the Cayley set $S = G - \{1\}$. Assign a unique color to each Cayley set C_g in $\mathcal{A}$. In the complete Cayley graph $\mathrm{Cay}(G, S)$, we color each edge according to the color associated with its Cayley set in $\mathcal{A}$. We call resulting graph the *complete colored Cayley graph* associated to a group G.

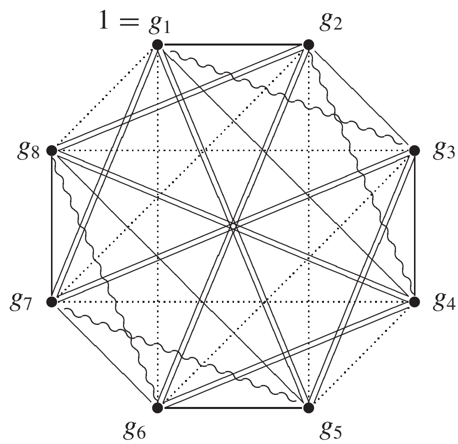

Figure 2 An example of a complete colored Cayley graph

Given the list $\{1 = g_1, g_2, \dots , g_n\}$ of elements G (with the identiy specified) and the complete colored Cayley graph of G, to what extent can we reconstruct the multiplication table of G? That is the question that we will address in this article.

Determining the multiplication table FIGURE 2 shows an example of a complete colored Cayley graph, where different line styles indicate the colors of edges. What, if anything, does this complete colored Cayley graph tell us about the group operation? The identity element, $1 \in G$, is given in the statement of the problem itself when we assume G is given as the set $G = \{1 = g_1, g_2, g_3, \dots , g_n\}$. Using this, we may identify any Cayley set as the set of elements immediately adjacent to the identity via edges of the same color. For example in FIGURE 2, the Cayley sets are $\{g_3\}$, $\{g_2, g_4\}$, $\{g_5, g_7\}$, $\{g_6, g_8\}$. The involutions of the group (elements of order two) are the only elements that appear in singleton Cayley sets, so these elements of G are determined by the complete colored Cayley graph. Since for any element $g \in G$ of order larger than two, g^{-1} is the only element paired with g in doubleton Cayley set, the complete colored Cayley graph determines inverses. In particular, in FIGURE 2, g_3 is the only involution, $g_2^{-1} = g_4$, and the solid edges represent multiplication by $g_2^{\pm 1}$. By following these solid edges we see that for $a = g_2$, we have $a^2 = g_3$, and $a^3 = g_4 = a^{-1}$. An easy induction argument shows that, for any $a \in G$, the powers of a are determined by the complete colored Cayley graph. We will begin tackling our key question, "What can we say about the multiplication table for this group?", by determining the powers of the elements given in the vertex set.

We've seen that element g_2 generates a subgroup of order 4, and we observe that the same is true of g_8 and of g_5 by following the dotted and "=" edges respectively. We also note that $g_2^2 = g_3 = g_8^2 = g_5^2$, $g_2^3 = g_4$, $g_8^3 = g_6$, and $g_5^3 = g_7$. As noted before, there is one element, g_3, which has order two.

The wavy edge corresponds to multiplication by the involution, g_3. Geometrically this means that there is only one wavy edge emanating from a vertex, so following that edge, say, leading from g_2 to g_4 tells us that $g_2g_3 = g_4$. This illustrates a general principle: multiplication by an involution is never ambiguous, but is always determined by the Cayley data.

Things get more complicated when we try to compute g_2g_8. Right multiplication by g_8 is represented by dotted edges. There are two vertices that we can reach from g_2 via dotted edges, g_5 and g_7, so we conclude that either $g_2g_8 = g_5$ or $g_2g_8 = g_7$. Similarly, following the double edges we see that there are two possibilities, $g_2g_5 = g_8$ or $g_2g_5 = g_6$. These possibilities are tabulated in TABLE 1.

TABLE 1: An attempt at filling in the multiplication table

$*$	1	g_2	g_4	g_8	g_6	g_5	g_7	g_3
1	1	g_2	g_4	g_8	g_6	g_5	g_7	g_3
g_2	g_2	g_3	1	g_5 *or* g_7	g_7 *or* g_5	g_6 *or* g_8	g_8 *or* g_6	g_4
g_4	g_4	1	g_3	g_7 *or* g_5	g_5 *or* g_7	g_8 *or* g_6	g_6 *or* g_8	g_2
g_8	g_8	g_7 *or* g_5	g_5 *or* g_7	g_3	1	g_2 *or* g_4	g_4 *or* g_2	g_6
g_6	g_6	g_5 *or* g_7	g_7 *or* g_5	1	g_3	g_4 *or* g_2	g_2 *or* g_4	g_8
g_5	g_5	g_8 *or* g_6	g_6 *or* g_8	g_4 *or* g_2	g_2 *or* g_4	g_3	1	g_7
g_7	g_7	g_6 *or* g_8	g_8 *or* g_6	g_2 *or* g_4	g_4 *or* g_2	1	g_3	g_5
g_3	g_3	g_4	g_2	g_6	g_8	g_7	g_5	1

Using reasoning that will be familiar to anyone who has worked on sudoku puzzles, where each symbol can appear exactly once in each row and each column, we can go farther. Suppose $g_2g_8 = g_5$; then necessarily $g_2g_6 = g_7$. Further, $g_2g_5 = g_2g_2g_8 = g_3g_8 = g_6$, and then it must follow that $g_2g_7 = g_8$. This completes the g_2 row. Multiplication in the g_4 column may be filled in by noticing that $g_4 = g_2^{-1}$. This, along with $(g_8g_4)^{-1} = g_2g_6 = g_7$, as was determined previously, tells us that $(g_8g_4) = g_7^{-1} = g_5$. Continuing in this way, the assumption that $g_2g_8 = g_5$ allows us to complete the entire multiplication table.

The reader should verify that the alternative possibility that $g_2g_8 = g_7$ allows the completion of a different multiplication table—one that is the transpose of the first. Which table is the correct one? There does not appear to be any way to resolve these questions, but is this because there is not enough information in the complete colored Cayley graph, or because we have not been clever enough? For which groups, if any, can we do better? Moreover, is this the worst that could happen? Will we always be able to determine the table for the group operation, up to its transpose?

Our second example is shown in FIGURE 3. Here, and later on, it will be convenient to draw the colored Cayley graph fragments separately. We will refer to such a collection as the *complete collection* of colored Cayley graphs.

FIGURE 3 shows the complete collection of colored Cayley graphs for S_3, the symmetric group on three elements. Let's review what we've learned about mining these graphs for information. The identity element is given. The elements immediate adjacent to the identity 1 are the elements of the associated Cayley set, so we see immediately that the Cayley sets are $\{g_2, g_3\}$, $\{g_6\}$, $\{g_4\}$, and $\{g_5\}$, respectively, in clockwise order from the upper left-hand graph. By following successive edges in different Cayley graphs, one can see, for example, that the product $g_2g_5 = g_4$, because g_4 is adjacent to g_2 in the graph whose Cayley set is $\{g_5\}$, and so on.

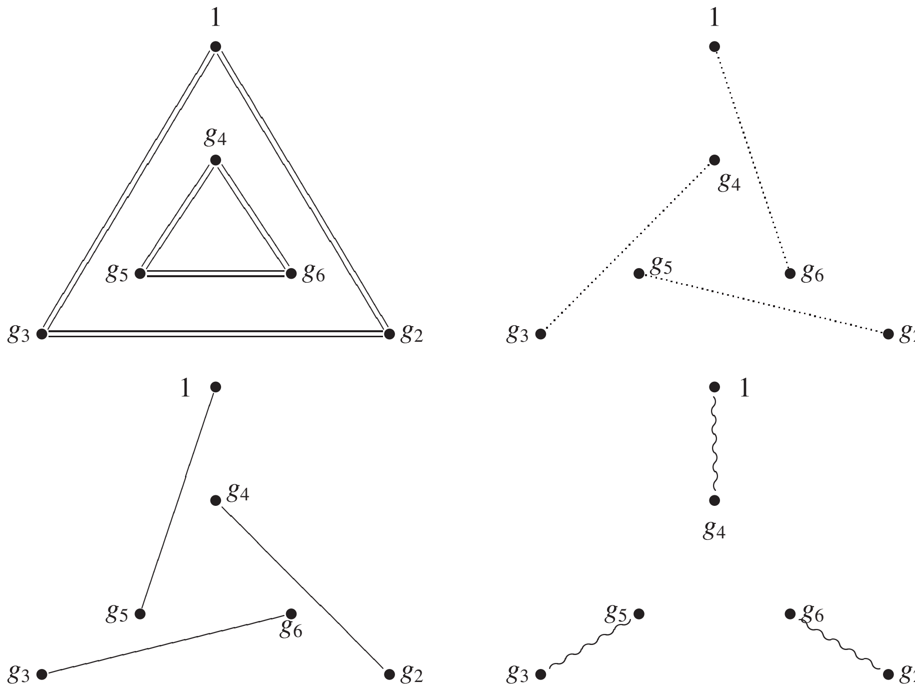

Figure 3 The colored Cayley graphs for S_3

We can't calculate g_4g_2 directly, because multiplication by g_2 and its inverse is represented by double edges, and g_4 is adjacent to both g_5 and g_6 in the Cayley graph with double edges. Either $g_4g_2 = g_5$ or $g_4g_2 = g_6$. Our prior experience suggests examining the inverse, and indeed, $(g_4g_2)^{-1} = g_2^{-1}g_4^{-1} = g_3g_4$, and the latter product is easy to calculate in the Cayley graph with wavy edges, corresponding to the Cayley set $\{g_4\}$, leading to the conclusion that $g_4g_2 = g_5^{-1} = g_5$.

The correspondence $g_2 = (123),\ g_3 = (231),\ g_4 = (23),\ g_5 = (13)$ and $g_6 = (12)$ assigns the elements in the vertex set their traditional meaning as permutations. It is worth taking the time needed to convince yourself that, in this case, the entire multiplication table is determined by the complete collection of colored Cayley graphs, without resorting to any of the information given above concerning the representation of any element g_i as a particular permutation.

Here is a summary the general method: to compute ab from the Cayley data, identify the color of edges associated with multiplication by $b^{\pm 1}$. Using these edges alone we focus on the set of elements adjacent to a, called the *neighborhood of* a, $N(a)$, in $\mathrm{Cay}\big(G, \{b, b^{-1}\}\big)$. If $b = b^{-1}$, then $N(a) = \{ab\}$, and ab has been computed. If $b \neq b^{-1}$, then $N(a) = \{v, w\}$, one of v and w is ab, and one is ab^{-1}, and the whole problem revolves around distinguishing which is which. Now we can formulate our two questions:

QUESTION 1. Does the complete colored Cayley graph for a group G suffice to determine the multiplication table for G?

It turns out that that the answer to question 1 is usually yes, but not always! A group whose multiplication table cannot be determined from its complete colored Cayley graph will be called an *ambiguous group*. We will be able to classify these. The answer will involve the idea of an *opposite group*, hence the title of this article. We will see that ambiguous groups are groups that cannot be distinguished from their opposites using the complete colored Cayley graph. We will define the opposite group a little later.

Before we move on, we pose our second question:

QUESTION 2. Does the complete colored Cayley graph for a group G suffice to determine the isomorphism class for the group?

The answer is surprising, and we postpone it until the very end of the paper.

The process of answering these questions affords a glimpse into the work of three mathematicians: the Irish mathematician, physicist, and astronomer, Sir William Rowan Hamilton (1805–1865); the German algebraist and number theorist Richard Dedekind (1831–1916); and the German algebraist Reinhold Baer (1902–1979).

Opposite groups and their graphs For any group, the transpose of the table for the binary operation defines an operation that satisfies all of the group axioms, but, unless G is abelian, this new operation is different from the original operation. Because the multiplication is accomplished in the opposite order from the multiplication in G, mathematicians call this new group, $G^\bullet$, the *opposite group*. Denoting multiplication in $G^\bullet$ by $\bullet$, we have $a \bullet b = ba$. Observe first that, if G is nonabelian, then G and $G^\bullet$ have different operations, and second that the inversion map, $\phi : g \mapsto g^{-1}$, is an isomorphism from G to $G^\bullet$. Moreover this map fixes Cayley sets, that is $\phi(S) = S$ whenever S is a Cayley set of G.

Since for all $g \in G$, $gg^{-1} = g^{-1}g = 1_G$, any Cayley set S for G must be a Cayley set for $G^\bullet$. It is natural to compare Cay(G, S) with Cay$(G^\bullet, S)$. FIGURE 4 exhibits colored Cayley graphs for a group and its opposite group. The example uses the symmetric group on three elements, and Cayley set $\{(123), (231), (23)\}$.

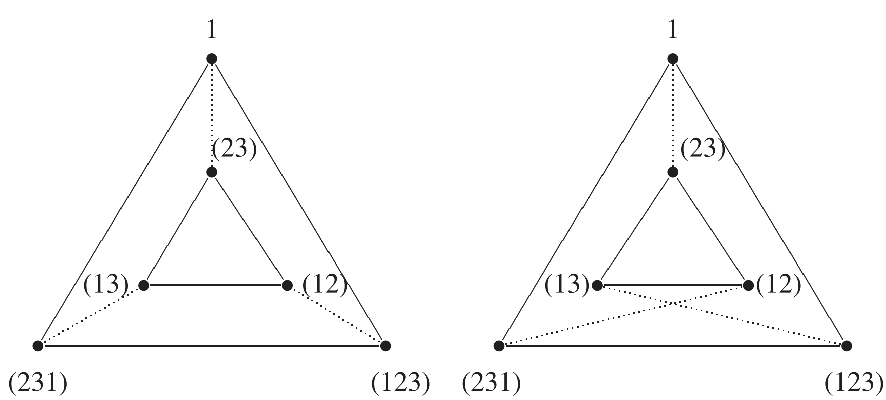

Figure 4 Two colored Cayley graphs, one for a group and the other for its opposite

You will note that in the left-hand graph (13) is adjacent via dotted edges to (231), while this pair is not adjacent in the right-hand graph. Instead, in the right hand graph, (13) is adjacent via dotted edges to $(231)^{-1}$. The reader should verify that the inversion map ϕ fixes Cayley sets, and that it induces a graph isomorphism $\phi : \text{Cay}(G, S) \cong \text{Cay}(G^\bullet, S)$. This illustrates the rather subtle difference: like the groups, the graphs are not ordinarily equal, but are always isomorphic.

Although the Cayley graphs for G and $G^\bullet$ are different, it is nevertheless true that the complete colored Cayley graph for G determines the complete colored Cayley graph for $G^\bullet$. First, it is easy to verify that S is a Cayley set of G if and only if S is a Cayley set of $G^\bullet$. Second, the vertices of Cay(G, S) and the vertices of Cay$(G^\bullet, S)$ are the same set, G. Third, the fact that inversion is a graph isomorphism between the graphs $\mathrm{Cay}(G, S)$ and $\mathrm{Cay}(G^\bullet, S)$ tells us that $a \sim b$ in $\mathrm{Cay}(G, S)$ if and only if $b^{-1} \sim a^{-1}$ in $\mathrm{Cay}(G^\bullet, S)$. Since both the edges of $\mathrm{Cay}(G, S)$, and all the inverse pairs are given as part of the complete colored Cayley graph, the edges of $\mathrm{Cay}(G^\bullet, S)$ are determined as well. The reader may want to use this principle and FIGURE 3 to construct the complete colored Cayley graph for ${S_3}^\bullet$.

Ambiguity and the quaternions A simple algebraic formulation will help us characterize the groups G for which $\mathrm{Cay}(G, S) = \mathrm{Cay}(G^\bullet, S)$. We call a group G a *balanced group* if, for all $a, b \in G$, either $ab = ba$ or $a^2 = b^2$. Every abelian group is balanced, but for a more interesting example of a balanced group, we consider quaternions. A group H is called a *quaternion group* if H is isomorphic to a group with presentation

$$\left\langle a,\ b \mid a^4 = 1 = b^4,\ a^2 = b^2,\ bab^{-1} = a^{-1} \right\rangle.$$

The correspondence $a = i$, $b = j$, $ab = k$ represents H as the familiar group Q_8 on the set $\{\pm 1, \pm i, \pm j, \pm k\}$ with group operation determined by $ij = k$, $jk = i$, $ki = j$, $kj = -i$, $ji = -k$, $i^2 = j^2 = k^2 = -1$. We have seen the Cayley graphs for the quaternion group already: the graph we discussed in the example of FIGURE 2 represents the quaternion group. The reader may want to verify this fact, using the assignment $i = g_2$, $j = g_8$, $k = g_5$.

The quaternion group was discovered in 1843 by Sir William Rowan Hamilton, while he was searching for a way to, as he put it, "multiply vectors." In a letter to his son Archibald, he reported that the idea came to him in a flash on the 16th of October in 1843, and in his excitement he was unable to " ... resist the impulse—unphilosophical as it may have been—to cut with a knife on a stone of Brougham Bridge, as we passed it, the fundamental formula with the symbols, i, j, k ; namely, $i^2 = j^2 = k^2 = ijk = -1$, which contains the Solution of the Problem ... " (The entire text of this letter, available on-line, is worth reading [**10**].)

It is easy to verify that the quaternion group is balanced, and we invite the reader to do so. The property of being balanced is important, because if a group G is balanced, then each Cayley graph of G is actually *equal*, as a graph, to the corresponding Cayley graph of the opposite group $G^\bullet$. The proof is not difficult, since we need only show that, in a balanced group, $a \sim b$ if and only if $a \overset{\bullet}{\sim} b$, where $\overset{\bullet}{\sim}$ denotes adjacency in the opposite graph. We leave the proof as an exercise for the reader.

Now, note that if G is nonabelian, then the group operation table for G is different from the table for $G^\bullet$. If, in addition, the complete colored Cayley graph for G is equal to that of its opposite group, then G must be ambiguous. Consequently, any balanced, nonabelian group is ambiguous. In particular, the quaternion group is ambiguous! This explains our difficulties in constructing the multiplication table from FIGURE 2. We could not differentiate between the table for G and the transposed table, the table for $G^\bullet$. Note the contrast with our experience with S_3. We saw from FIGURE 4 that the colored Cayley graph for ${S_3}^\bullet$ was different from that of S_3, and we were successful in recovering the multiplication table for S_3 from the complete colored Cayley graph. This is not accidental. In order for G to be ambiguous, there must be elements $a, b \in G$ such that we cannot determine ab from the complete colored Cayley graph for G. In this case, we also say that *the product ab is ambiguous*. We will prove that the only way that this may happen is when the subgroup generated by the elements a and b is isomorphic to the quaternion group.

THE SUBGROUP THEOREM. *If G is an ambiguous group, then there is a subgroup H in G that is isomorphic to the quaternion group.*

Suppose that a and b have an ambiguous product in G. Note that we may assume neither a nor b is involution, because the product of any element g by an involution x is determined by the singleton neighborhood $N(g)$ in $\mathrm{Cay}(G, \{x\})$, for multiplication by x on the right, and in the opposite graph for multiplication on the left. So when ab is an ambiguous product, both $\{a, a^{-1}\}$ and $\{b, b^{-1}\}$ are Cayley sets containing two distinct elements.

We will first establish three consequences of the ambiguity of the product ab. These facts are inherently appealing consequences of the geometry of the Cayley graphs.

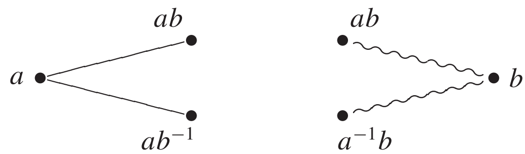

Figure 5 Comparing $N(a)$ in $\mathrm{Cay}\big(G, \{b, b^{-1}\}\big)$ with $N(b)$ in $\mathrm{Cay}\big(G^{\bullet}, \{a, a^{-1}\}\big)$.

CONSEQUENCE 1. *If ab is ambiguous, then $a^2 = b^2$.*

Look at the two Cayley graph fragments in FIGURE 5. In this figure, the left fragment occurs in $\mathrm{Cay}\big(G, \{b, b^{-1}\}\big)$, while the right fragment occurs in the opposite graph, $\mathrm{Cay}\big(G^{\bullet}, \{a, a^{-1}\}\big)$. We have used the fact that the complete colored Cayley graph for G determines the colored Cayley graphs for $G^{\bullet}$. Our labels of ab, ab^{-1}, and $a^{-1}b$ on vertices are correct, since we know the contents of the neighborhoods of a and b, but in an actual graph we may not know which vertex goes with which label.

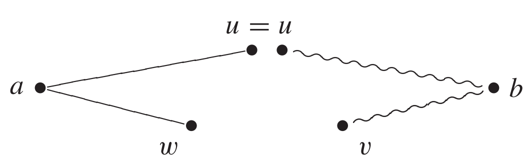

Figure 6 A better picture.

What we really see are the neighborhoods of a and of b. These are depicted in FIGURE 6. Because neither a nor b is an involution, each neighborhood is a doubleton. The element in common to both neighborhoods must be ab, unless the neighborhoods intersect in two elements, in which case $v = w$, or $ab^{-1} = a^{-1}b$. Since ab is ambiguous, the latter case must hold, and neighborhoods must be equal. We conclude that $ab^{-1} = a^{-1}b$, which implies that $a^2 = b^2$.

CONSEQUENCE 2. *If ab is ambiguous, then $a^4 = b^4 = 1$.*

We examine the two graph neighborhoods illustrated in FIGURE 7.

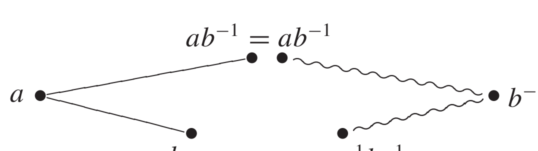

Figure 7 Comparing $N(a)$ in $\mathrm{Cay}\big(G, \{b, b^{-1}\}\big)$ with $N(b^{-1})$ in $\mathrm{Cay}\big(G^{\bullet}, \{a, a^{-1}\}\big)$.

In FIGURE 7, the left fragment uses the Cayley set $\{b, b^{-1}\}$, while the right fragment uses $\{a, a^{-1}\}$ in the opposite graph. This time, the element ab is the single element in

$N(a)$ that is not an element of $N(b^{-1})$. But the ambiguity of ab implies there is no such single element, hence we conclude that $N(a) = N(b^{-1})$. But then, $ab = a^{-1}b^{-1}$ which implies $a^2 = b^{-2} = b^2$, from which it follows that $b^4 = 1 = a^4$.

CONSEQUENCE 3. *If ab is ambiguous, then $b^{-1}ab = a^{-1}$.*

Consider the neighborhood $N(a) = \{u, v\}$ of a in $\text{Cay}(G, \{b, b^{-1}\})$. In this Cayley graph $N(a)$ contains the ab and the element ab^{-1}. Our problem is to decide which of the elements u or v is ab. We do this by reasoning indirectly. We observe that $a(ab) = (a^2)b = (b^2)b = b^{-1}$, so that $b^{-1} \in N(a)$ in the Cayley graph $\text{Cay}(G, \{ab, (ab)^{-1}\})$. This leads us to compare the neighborhood $N(a)$ in $\text{Cay}(G, \{u, u^{-1}\})$, with $N(a)$ in $\text{Cay}(G, \{v, v^{-1}\})$. Assume, without loss of generality, that $ab = u$, and $ab^{-1} = v$.

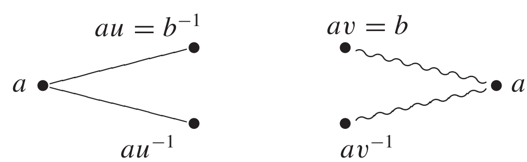

Figure 8 Comparing the neighborhoods Cay(G, $\{u, u^{-1}\}$) and Cay(G, $\{v, v^{-1}\}$).

We refer to the graph fragments in FIGURE 8, where the left fragment is from $\text{Cay}(G, \{u, u^{-1}\})$ and the right fragment is from $\text{Cay}(G, \{v, v^{-1}\})$. Since $b^{-1} = b^3$ and $b^2 = a^2$, we can deduce that $au = a(ab) = b^{-1}$ and $av = a(ab^{-1}) = b^2b^{-1} = b$. This will determine the product ab: It is the element u, such that $N(a)$ in $\text{Cay}(G, \{u, u^{-1}\})$ contains b^{-1}, as opposed to b itself. But ab is ambiguous, so determination by this method must be flawed. In particular, it must be true that both neighborhoods are exactly the Cayley set $\{b, b^{-1}\}$. We conclude that $a(b^{-1}a^{-1}) = b^{-1}$, and that $a(ba^{-1}) = b$. These two statements are equivalent, and each implies that $b^{-1}a^{-1} = a^{-1}b$, from which we see that $b^{-1}ab = ab^2 = a^3 = a^{-1}$.

Now we can prove the Subgroup Theorem.

Proof. Suppose ab is an ambiguous product, and consider the subgroup H generated by $\{a, b\}$. Our previous analysis of the Cayley fragments showed us that $a^2 = b^2$, $a^4 = b^4 = 1$, and that $b^{-1}ab = a^{-1}$. In addition, we might note that $b \notin \langle a \rangle$, otherwise ab would be a power of a, and ab would not be ambiguous.

We claim that $\langle a \rangle$ is a normal subgroup of H. Because H is generated by a and b, it suffices to check conjugation by b. But we already know that $b^{-1}ab = a^{-1} \in \langle a \rangle$. Thus $\langle a \rangle$ is a normal subgroup of H.

Because $b^2 = a^2 \in \langle a \rangle$, the subgroup H consists of exactly two cosets: $\langle a \rangle$ and $\langle a \rangle b$. Thus $H = \{1, a, a^2, a^3\} \cup \{b, ab, a^2b, a^3b\}$. Since $b^{-1}ab = a^{-1}$, H is nonabelian.

Now, H is a group of order 8, and, up to isomorphism, there are two nonabelian groups of order 8: the dihedral group D_8 and the quaternion group Q_8. The dihedral group D_8 has exactly 2 elements of order 4. In contrast, H has at least 3 elements of order 4, namely a, a^{-1}, and b. We conclude that $H \ncong D_8$ and that therefore $H \cong Q_8$. ■

Examples, Hamiltonian groups, and the larger picture

More may be said. We can prove a classification theorem for ambiguous groups that tells us exactly what the structure of an ambiguous group must be. The answer involves groups in which every element except the identity element has order two, so that if the group is finite it must be isomorphic to $\underbrace{Z_2 \times \cdots \times Z_2}_{n\ times}$, $n \geq 1$. Such a group is called called an *elementary abelian two-group*.

THE CLASSIFICATION THEOREM. *Every ambiguous group G is isomorphic to Q_8 or to a direct product of the quaternion group Q_8 and an elementary abelian two-group A. Conversely, every direct product $Q_8 \times A$ is ambiguous. Multiplication in the group A is determined by the complete colored Cayley graph for G. There are exactly two multiplications for the set G that are consistent with the complete colored Cayley graph, that of G and that of $G^\bullet$.*

Readers interested in the details of the proof of the Classification Theorem should consult [**12**]. The Classification Theorem gives us instant access to a method for differentiating between ambiguous and nonambiguous groups: The following are ambiguous: Q_8, $Q_8 \times Z_2$, $Q_8 \times \mathbb{Z}_2 \times Z_2$, $Q_8 \times Z_2 \times Z_2 \times Z_2 \times Z_2 \times \cdots \times Z_2$. But $Q_8 \times Z_4$, $Q_8 \times D_8$, $Q_8 \times Z_{17}$, and $Q_8 \times Q_8$ are not ambiguous!

You may find the above examples puzzling. How exactly can one resolve the apparent ambiguity of $(i, 1)(j, 1)$ in $Q_8 \times Q_8$? The following example illustrates how an unambiguous multiplication might be used to determine a product in an unambiguous group in which Q_8 is a direct factor. In order to communicate the key idea, we'll make the simplifying assumption that the elements of our groups are given as ordered pairs.

Let K be any group containing an element c of order four. How might we use a Cayley graph method to determine the product $(i, 1)(j, 1)$ from the complete colored Cayley graph for $Q_8 \times K$? The product $(i, 1)(j, 1) = (ij, 1)$. The ambiguity in question is that of determining the correct product ij in component one. Because the complete colored Cayley graph for a group determines the order of any element in G, we may unambiguously choose an element $(1, c) \in Q_8 \times K$ of order 4. The element $(1, c^2)$ is an involution of $Q_8 \times K$, and is determined by the complete colored Cayley graph because it is a power of the element $(1, c)$. Consider the products $(i, 1)(1, c)$ and $(j, 1)(1, c)$. These products are unambiguous, because multiplication of any element by 1 on either side is unambiguous. Using the element(j, c) produced by the second multiplication, we may identify the Cayley set $S = \{(j, c), (-j, c^{-1})\}$, and its corresponding colored Cayley graph. The neighborhood $N\big((i, c)\big)$ using Cayley set S is shown below in FIGURE 9.

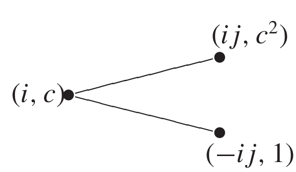

Figure 9 Resolving apparent ambiguity in $Q_8 \times K$

The correct product ij will be found in the first component of the element in $N\big((i, c)\big)$ whose second component is the involution c^2 of K. We may identify c^2, because it is the second component of the element $(1, c)^2$. This resolves the apparent ambiguity.

A group with the property that every subgroup is normal is called *Hamiltonian*. Dedekind, perhaps best known for his use of the Dedekind cut to prove the completeness of the reals without the axiom of choice, and credited with the definition of the term *ideal*, named these groups in honor of William Rowan Hamilton, the bridge-carving discoverer of the quaternions. Dedekind's discovery, in 1887, explains the rationale for the name: Every nonabelian Hamiltonian group must contain a subgroup isomorphic to a quaternion group [**5**]. Nearly fifty years later, in 1933, the German mathematician Reinhold Baer completed the classification of nonabelian Hamiltonian groups [**1**]:

BAER'S THEOREM. *A nonabelian group G is a Hamiltonian group if and only if $G = Q_8 \times A \times B$, where A is an elementary abelian two-group and B an abelian group with every element of odd order.*

Comparing ambiguous groups with Hamiltonian Groups, and appealing to the Classification Theorem, we find that the ambiguous groups are precisely the nonabelian Hamiltonian groups with no elements of odd order.

EXERCISE 1. Modify the technique used above to resolve ambiguity in $Q_8 \times K$ (using FIGURE 9) to find a complete colored Cayley graph method to determine the product of $(i, 0)(j, 0)$ in $Q_8 \times Z_5$, a group that is Hamiltonian, but not ambiguous.

This brings us to our second question. Does the complete colored Cayley graph for G determine its isomorphism class? If G is unambiguous, then by definition, the multiplication table is determined, and so we know the isomorphism class of the group.

According to the classification theorem, any ambiguous group G has the form $Q_8 \times A$, where A is abelian, and where for all $a \in A$, $a^2 = 1_A$. Consider elements (g, a) and (h, b) in G, with $g, h \in Q_8$ and $a, b \in A$. If $gh = hg$ then (g, a) commutes with (h, b). Otherwise, since Q_8 is balanced, $g^2 = h^2$, and hence $(g, a)^2 = (g^2, 1_A) = (h^2, 1_A) = (h, b)^2$, so G is balanced. Therefore, the complete colored Cayley graph for G is the same as that of $G^\bullet$, and so the ambiguous groups are precisely those nonabelian groups that cannot be distinguished from their opposites using the complete colored Cayley graph.

Because the complete colored Cayley graph for G determines the data for $G^\bullet$, it determines whether the complete colored Cayley graphs for G and for $G^\bullet$ are identical. Suppose G is such a group. Then G is ambiguous, and there are exactly two possible tables for G, that of G itself and that of $G^\bullet$. Either one of these tables produces a group isomorphic to G, so the isomorphism class is for G is determined. This shows us that the answer to Question 2 is yes, the complete colored Cayley graph for the group does determine its isomorphism class. Given that we lost information about the direction of the edges when we moved from the directed Cayley digraph to the undirected Cayley graph, it may not be surprising that we encounter ambiguity. What is surprising is that the *only* information that is lost is the ability to differentiate between opposites.

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Summary For any finite group, if we are given the complete Cayley graph for the group, and if the undirected edges are colored in a natural way, does this partial information determine the multiplication table for the group? It turns out that that the answer to this inverse problem is usually yes, but not always. We call a group whose multiplication table cannot be determined from its complete colored Cayley graph an *ambiguous group*. A simple example of such a group is the quaternion group. We are able classify all ambiguous groups. We show that the complete Cayley graph with colored edges does determine the isomorphism class for the group. Along the way we revisit contributions made to the development of group theory by the eminent mathematicians Cayley, Hamilton, Dirichlet, and Baer.

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NOTES

Probability in Look Up and Scream

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In 2006 my wife and I accompanied our church's youth group on a mission trip to Traverse City, Michigan, where youth from the host church introduced us to the "Yell Game." Players stand in a circle, close their eyes, and on the count of three, open their eyes, looking directly at another player. If two players are looking directly at each other, they yell and are out of the game. We watched a game begin with about 25 people. For three consecutive rounds, nobody yelled; no two people ever looked up at each other.

What is the probability of this happening? To answer this question we need a probabilistic model. We will assume that each person is equally likely to look at any other person in the circle, and the looks are independent in the sense that knowing that Parker looks at Paige will not affect the probability that Linda (or Paige) looks at anyone else playing the game. (These assumptions might not be true in an actual game—especially if the group contains a boyfriend and girlfriend who long to gaze in each other's eyes!)

Some time later, while skimming journals that were circulating in my department, I discovered the article "Look Up and Scream: Analytical Difficulties Resolved!" written by S. P. Bhatia [**2**] which cited Balas and Connor for introducing the game, which they called "Look Up and Scream" [**1**]. In both of these articles, the authors assume an even number of players. Balas and Connor [**1**] show how to represent the game with a function, and give numerical results for the number of yells based on exhaustive searches. Bhatia [**2**] represents a round of the game with a digraph and gives a formula for the number of ways a round of the game can be played with no yells, and thus the probability that no yells occur. (In addition to the unconstrained game, both [**1**] and [**2**] consider a variation of the game in which each player must look at a player to the immediate right, immediate left, or across the circle. We do not address these variations here.)

In this paper we extend these results by allowing an arbitrary number of players, and we present a formula giving the number of ways a round can be played with y pairs of yells. Our derivation of the formula for no yells differs from the derivation in [**2**]. Finally, we derive formulas for the mean and variance of the number of pairs of yells in two different ways, and show how to calculate the mean number of rounds of an n-player Look Up and Scream game.

Inclusion-exclusion The principle of inclusion-exclusion tells us that the number of elements in the union of finite sets $A_1, A_2, \ldots, A_n$ is

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$$|A_1 \cup A_2 \cup \cdots \cup A_n| = \sum_{1\le i\le n} |A_i| - \sum_{1\le i<j\le n} |A_i \cap A_j| + \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| - \cdots + (-1)^{n+1}|A_1 \cap A_2 \cap \cdots \cap A_n|.$$

If S is a larger set containing the union of these sets, then the number of elements of S that appear in none of the sets $\{A_i\}$ is

$$|S| - |A_1 \cup A_2 \cup \cdots \cup A_n| = |S| - \sum_{1\le i\le n} |A_i| + \sum_{1\le i<j\le n} |A_i \cap A_j| - \sum_{1\le i<j<k\le n} |A_i \cap A_j \cap A_k| + \cdots + (-1)^{n}|A_1 \cap A_2 \cap \cdots \cap A_n|. \quad (1)$$

We use a generalization that counts the number of elements that appear in exactly m of the sets. Let $S_k = \sum |A_{i_1} \cap A_{i_2} \cap \cdots \cap A_{i_k}|$ where the sum is taken over all possible selections of k of the n sets. Then the number of elements that appear in exactly m of the sets is

$$E_m = S_m - \binom{m+1}{1} S_{m+1} + \binom{m+2}{2} S_{m+2} - \cdots + (-1)^{n-m}\binom{n}{n-m} S_n. \quad (2)$$

In particular if $m = 0$ and $S_0 = |S|$, then we have the number of elements that appear in no sets, so that (2) generalizes (1). For a proof of this result see [**4**, p. 400] or [**5**, pp. 276–277].

Combinatorial derivation of distribution of pairs of yells Let $A_{i,j}$ be the set of all rounds of n-player Look Up and Scream for which Player i and Player j look at each other. Then $A_{i_1,j_1} \cap A_{i_2,j_2} \cap \cdots \cap A_{i_k,j_k}$ is the set of rounds where Player i_1 yells at Player j_1, Player i_2 yells at Player j_2, … , and Player i_k yells at Player j_k. Note for this intersection to be nonempty, all of the subscripts must be unique, and thus k must be at most $\lfloor \frac{n}{2} \rfloor$. When all of the subscripts are unique, we already know at whom $2k$ of the players look, and the other $n - 2k$ players may look at anyone. Thus, $|A_{i_1,j_1} \cap A_{i_2,j_2} \cap \cdots \cap A_{i_k,j_k}| = (n-1)^{n-2k}$.

For a given k, how many of these intersections are there? To form one of these intersections, we first select our $2k$ players in $\binom{n}{2k}$ ways. Then we select two of the $2k$ players to match, select two of the remaining $2k - 2$ players to match, and continue to do this until the final two players are matched. Since we don't care about the order in which we create these matches, the number of ways to match these $2k$ players is $\frac{\binom{2k}{2}\binom{2k-2}{2}\cdots\binom{2}{2}}{k!}$. Thus, the number of ways to form one of these intersections of k sets is $\frac{\binom{n}{2k}\binom{2k}{2}\binom{2k-2}{2}\cdots\binom{2}{2}}{k!}$. Noticing that one of the factorials in each of the denominators of the binomial coefficients cancels with a factorial in the numerator of the next coefficient, we can simplify this expression to $\frac{n!}{(n-2k)!k!2^k}$. Thus,

$$S_k = \sum |A_{i_1,j_1} \cap A_{i_2,j_2} \cap \cdots \cap A_{i_k,j_k}| = \sum (n-1)^{n-2k} = \frac{n!}{(n-2k)!\,k!\,2^k}(n-1)^{n-2k}.$$

Let $D_{n,y}$ be the number of ways a round of n-player Look Up and Scream can result in exactly y pairs of yells. According to (2), $D_{n,y} = S_y - \binom{y+1}{1}S_{y+1} + \binom{y+2}{2}S_{y+2} -$

$\cdots + (-1)^{n-y}\binom{n}{n-y}S_n$; but for S_k to be nonzero, we must have $k \le \lfloor \frac{n}{2} \rfloor$, and thus

$$
\begin{aligned}
D_{n,y} &= S_y - \binom{y+1}{1}S_{y+1} + \binom{y+2}{2}S_{y+2} - \cdots + (-1)^{\lfloor \frac{n}{2} \rfloor - y}\binom{\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{n}{2} \rfloor - y}S_{\lfloor \frac{n}{2} \rfloor} \\
&= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - y} (-1)^j \binom{y+j}{j} S_{y+j} \\
&= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - y} (-1)^j \binom{y+j}{j} \frac{n!}{(n-2(y+j))!\,(y+j)!\,2^{y+j}}(n-1)^{n-2(y+j)} \\
&= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - y} (-1)^j \frac{n!}{(n-2(y+j))!\,y!\,j!\,2^{y+j}}(n-1)^{n-2(y+j)} \qquad (3)
\end{aligned}
$$

Using (3), we produce TABLE 1, the first row of which is sequence A134362 in [**6**].

TABLE 1: Rounds of n-player Look Up and Scream with y pairs of yells

	n players								
y	2	3	4	5	6	7	8	9	10
0	0	2	30	444	7360	138690	2954364	70469000	1864204416
1	1	6	48	520	7170	119826	2347072	52629984	1327962060
2	0	0	3	60	1080	20790	443100	10496304	275093280
3	0	0	0	0	15	630	20160	614880	19145700
4	0	0	0	0	0	0	105	7560	378000
5	0	0	0	0	0	0	0	0	945

Recalling our assumptions that each person is equally likely to look at anyone else and that these looks are independent, the probability of exactly y pairs of yells occurring in a round of n-player Look Up and Scream is simply

$$
P(n,y) = \frac{D_{n,y}}{(n-1)^n} = \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - y} (-1)^j \frac{n!}{(n-2(y+j))!\,y!\,j!\,2^{y+j}}(n-1)^{-2(y+j)}. \qquad (4)
$$

Returning to the question asked on the Traverse City mission trip, if 25 people play the game, then the probability of there being no matches for at least 3 consecutive rounds is $(P(25,0))^3 \approx (.58063)^3 \approx 0.196$, and thus this event was not too surprising.

Finally, the probability of playing a round of n-player Look Up and Scream and there being no yells is

$$
\begin{aligned}
P(n,0) &= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} (-1)^j \frac{n!}{(n-2j)!\,j!\,2^j}(n-1)^{-2j} \\
&= \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor} \left(\frac{-1}{2}\right)^j \cdot \frac{1}{j!} \cdot \frac{n(n-1)(n-2)\cdots(n-2j+1)}{(n-1)(n-1)(n-1)\cdots(n-1)}
\end{aligned}
$$

For any fixed value j, the ratio of factors involving n converges to 1 as n approaches infinity. Although in general we must be careful in interchanging limits and infinite sums, we may do this for the sum above and thus,

$$\lim_{n\to\infty} P(n,0) = \sum_{j=0}^{\infty} \left(\frac{-1}{2}\right)^j \cdot \frac{1}{j!} \cdot 1 = e^{-1/2} \approx 0.607.$$

Hence, approximately 60.7% of the time, no one will yell in a round of Look Up and Scream with many players.

Mean number of pairs of yells Let Y_n be the number of pairs of yells in a round of n-player Look Up and Scream. We claim that the mean number of pairs of yells is $E(Y_n) = \frac{n}{2(n-1)}$. This makes sense intuitively since the probability any particular person yells is $\frac{1}{n-1}$, and thus we expect approximately $\frac{n}{n-1}$ people to yell, which means we have $\frac{n}{2(n-1)}$ pairs of yells. More formally,

$$\begin{aligned} E(Y_n) &= \sum_{y=0}^{\lfloor \frac{n}{2} \rfloor} y \cdot P(n,y) \\ &= \sum_{y=0}^{\lfloor \frac{n}{2} \rfloor} y \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - y} (-1)^j \frac{n!}{(n-2(y+j))!\, y!\, j!\, 2^{y+j}} (n-1)^{-2(y+j)} \\ &= \frac{n}{2(n-1)} \sum_{y=1}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - y} (-1)^j \frac{(n-2)!}{(n-2(y+j))!\,(y-1)!\, j!\, 2^{y+j-1}} (n-1)^{-2(y+j-1)} \\ &= \frac{n}{2(n-1)} \sum_{\hat{y}=0}^{\lfloor \frac{n}{2} \rfloor - 1} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - \hat{y} - 1} (-1)^j \frac{(n-2)!}{(n-2-2(\hat{y}+j))!\, \hat{y}!\, j!\, 2^{\hat{y}+j}} (n-1)^{-2(\hat{y}+j)} \end{aligned}$$

We now change the index of the inner summation by letting $t = \hat{y} + j$ to get

$$E(Y_n) = \frac{n}{2(n-1)} \sum_{\hat{y}=0}^{\lfloor \frac{n}{2} \rfloor - 1} \sum_{t=\hat{y}}^{\lfloor \frac{n}{2} \rfloor - 1} (-1)^{t-\hat{y}} \frac{(n-2)!}{(n-2-2t)!\, \hat{y}!\, (t-\hat{y})!\, 2^t} (n-1)^{-2t}.$$

We are essentially summing over all $\{(\hat{y},t) | 0 \le \hat{y} \le t \le \lfloor \frac{n}{2} \rfloor - 1\}$. Interchanging the order of the summation, moving terms that do not depend upon $\hat{y}$ outside of the new inner sum, and introducing a fancy 1 (in the form of $\frac{t!}{t!}$), we get

$$E(Y_n) = \frac{n}{2(n-1)} \sum_{t=0}^{\lfloor \frac{n}{2} \rfloor - 1} \left(\frac{1}{2}\right)^t \left(\frac{1}{n-1}\right)^{2t} \frac{(n-2)!}{(n-2-2t)!\, t!} (-1)^t \sum_{\hat{y}=0}^{t} (-1)^{\hat{y}} \frac{t!}{\hat{y}!\,(t-\hat{y})!}.$$

Wearing our Binomial Theorem glasses, we recognize the inner sum as the expansion of $(1-1)^t$ and thus it is equal to 0 unless $t = 0$. When $t = 0$, $\hat{y}$ must also be 0, and thus we see the double summation is simply equal to 1 and $E(Y_n) = \frac{n}{2(n-1)}$.

Variance of the number of pairs of yells The variance of a random variable X is defined to be $\mathrm{Var}(X) = E((X-\mu)^2)$. A useful formula for finding the variance is $\mathrm{Var}(X) = E(X(X-1)) + E(X) - (E(X))^2$. We claim that the variance of the number of pairs of yells is $\mathrm{Var}(Y_n) = \frac{n(n-2)^2}{2(n-1)^3}$. First note,

$$E(Y_n(Y_n - 1)) = \sum_{y=0}^{\lfloor \frac{n}{2} \rfloor} y(y-1)P(n, y)$$

$$= \sum_{y=2}^{\lfloor \frac{n}{2} \rfloor} y(y-1) \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - y} (-1)^j \frac{n!}{(n-2(y+j))!\, y!\, j!\, 2^{y+j}} (n-1)^{-2(y+j)}$$

$$= \frac{n(n-2)(n-3)}{2^2(n-1)^3} \sum_{y=2}^{\lfloor \frac{n}{2} \rfloor} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - y} (-1)^j \frac{(n-4)!}{(n-2(y+j))!\,(y-2)!\, j!\, 2^{y+j-2}} (n-1)^{-2(y+j-2)}$$

$$= \frac{n(n-2)(n-3)}{4(n-1)^3} \sum_{\hat{y}=0}^{\lfloor \frac{n}{2} \rfloor - 2} \sum_{j=0}^{\lfloor \frac{n}{2} \rfloor - \hat{y} - 2} (-1)^j \frac{(n-4)!}{(n-4-2(\hat{y}+j))!\, \hat{y}!\, j!\, 2^{\hat{y}+j}} (n-1)^{-2(\hat{y}+j)} \quad (5)$$

Repeating the techniques from the last section, we can show $E(Y_n(Y_n - 1)) = \frac{n(n-2)(n-3)}{4(n-1)^3}$ by showing the double sum in (5) equals 1. Hence,

$$\begin{aligned}
\mathrm{Var}(Y_n) &= E(Y_n(Y_n - 1)) + E(Y_n) - (E(Y_n))^2 \\
&= \frac{n(n-2)(n-3)}{4(n-1)^3} + \frac{n}{2(n-1)} - \frac{n^2}{4(n-1)^2} \\
&= \frac{n(n-2)(n-3) + 2n(n-1)^2 - n^2(n-1)}{4(n-1)^3} \\
&= \frac{n(2n^2 - 8n + 8)}{4(n-1)^3} = \frac{n(n-2)^2}{2(n-1)^3}
\end{aligned}$$

Mean rounds in *n*-player Look Up and Scream Since we can calculate the probability of y pairs of yells in n-player Look Up and Scream using (4), we can model the game with a Markov chain. In fact, since the parity of the number of players never changes, we will model the game with two Markov chains based on the parity. We show how to model the game for even $n \leq 10$, assuming the game ends when there are 0 players left, leaving odd and larger even values of n for the reader.

The transition matrix for this Markov chain with 6 states {0,2,4,6,8,10} is a 6×6 matrix where the entry in row i and column j is the probability that a game with $2(i-1)$ players at the start of one round will have $2(j-1)$ players at the completion of the round. Thus, the transition matrix is

$$\begin{pmatrix}
1 & 0 & 0 & 0 & 0 & 0 \\
1 & 0 & 0 & 0 & 0 & 0 \\
\frac{3}{81} & \frac{48}{81} & \frac{30}{81} & 0 & 0 & 0 \\
\frac{15}{15625} & \frac{1080}{15625} & \frac{7170}{15625} & \frac{7360}{15625} & 0 & 0 \\
\frac{105}{5764801} & \frac{20160}{5764801} & \frac{443100}{5764801} & \frac{2347072}{5764801} & \frac{2954364}{5764801} & 0 \\
\frac{945}{3486784401} & \frac{378000}{3486784401} & \frac{19145700}{3486784401} & \frac{275093280}{3486784401} & \frac{1327962060}{3486784401} & \frac{1864204416}{3486784401}
\end{pmatrix}$$

To compute the mean rounds of an n player game, we remove the first row and column of the matrix above and call the new matrix A. Next we use MAPLE to find $B = (I - A)^{-1}$ where I is the 5×5 identity matrix:

$$B = \begin{pmatrix} 1 & 0 & 0 & 0 & 0 \\ \frac{16}{17} & \frac{27}{17} & 0 & 0 & 0 \\ \frac{8872}{9367} & \frac{12906}{9367} & \frac{3125}{163} & 0 & 0 \\ \frac{3559775872}{3760766197} & \frac{5269044276}{3760766197} & \frac{1047800000}{663664623} & \frac{823543}{401491} & 0 \\ \frac{42787599538135408}{45201066366790867} & \frac{63236169861007284}{45201066366790867} & \frac{12863607475700000}{7976658770610153} & \frac{8100991546508}{4825564894501} & \frac{129140163}{60095555} \end{pmatrix}$$

The entry of B in the row i and column j is the expected number of rounds with $2j$ players of a game beginning with $2i$ players. For example, a game beginning with 4 players is expected to have $\frac{16}{17}$ rounds with 2 players. Thus, if we sum the entries in row i, we get the expected number of rounds of a game beginning with $2i$ players. TABLE 2 gives the expected number of rounds of a game beginning with 2 to 10 players, rounded to the nearest hundredth.

TABLE 2: Expected number of rounds of Look Up and Scream

# of players	2	3	4	5	6	7	8	9	10
Expected # of rounds	1	1.33	2.53	2.96	4.22	4.69	5.98	6.48	7.79

Alternative derivation of mean and variance In undergraduate probability courses, students usually learn the formulas for the mean and variance of a linear combination of *independent* random variables. Namely, if $X_1, X_2, \dots, X_n$ are n independent random variables with means $\mu_1, \mu_2, \dots, \mu_n$ and variances $\text{Var}(X_1), \text{Var}(X_2), \dots, \text{Var}(X_n)$ and $Y = \sum_{i=1}^{n} a_i X_i$, then $\mu_Y = \sum_{i=1}^{n} a_i \mu_i$ and $\text{Var}(Y) = \sum_{i=1}^{n} {a_i}^2 \text{Var}(X_i)$. Although the formula for the mean holds if the X_i's are not independent, the formula for the variance does not hold. The more general formula for the variance is $\text{Var}(Y) = \sum_{i=1}^{n} {a_i}^2 \text{Var}(X_i) + 2 \sum\sum_{i<j} a_i a_j \text{Cov}(X_i, X_j)$ where $\text{Cov}(X_i, X_j)$ is the covariance of X_i and X_j and is defined to be $\text{Cov}(X_i, X_j) = E((X_i - \mu_i)(X_j - \mu_j))$ (see [**3**, p. 421]).

Play a round of n-player Look Up and Scream and let X_i equal 1 if Player i yells and 0 if Player i does not yell. Then the number of pairs of yells Y_n equals $\frac{X_1+X_2+\cdots+X_n}{2}$. Note the probability person i yells is $\frac{1}{n-1}$, the probability that the person at whom person i looks decides to look at person i. Thus, each X_i is a Bernoulli random variable with probability of success $p = \frac{1}{n-1}$, and thus mean $\frac{1}{n-1}$ and variance $\frac{1}{n-1}\left(1 - \frac{1}{n-1}\right) = \frac{n-2}{(n-1)^2}$. Using this relationship between Y_n and the X_i's, we can easily derive the mean number of pairs of yells to be

$$\begin{aligned} E(Y_n) &= E\left(\frac{X_1 + X_2 + \cdots + X_n}{2}\right) \\ &= \frac{1}{2}\big(E(X_1) + E(X_2) + \cdots + E(X_n)\big) = \frac{n}{2(n-1)}. \end{aligned}$$

In order to derive the formula for the variance using this method, note that the X_i's are not independent (for if we know that Player 1 yells and hence $X_1 = 1$, then it can't be the case that $X_i = 0$ for all $i \neq 1$). Thus, we must find $\text{Cov}(X_i, X_j)$ and we will use the shortcut formula $\text{Cov}(X_i, X_j) = E(X_i X_j) - \mu_i \mu_j$ to find the covariance. Now note $X_i X_j = 0$ unless both Player i and Player j yell in which case $X_i X_j = 1$. To find the probability that both Player i and Player j yell, we consider two cases, depending on whether Player i and Player j look at each other. Given our assumption that the looks are independent, the probability Player i and Player j look at each other is $(\frac{1}{n-1})^2$. The probability that Player i and Player j yell and are not looking at each other is the probability Player i looks at a Player k distinct from Player j, multiplied by the probability Player k looks at Player i, multiplied by the probability Player j looks at a Player l distinct from Players i and k, multiplied by the probability Player l looks at Player j, or $\frac{n-2}{n-1} \cdot \frac{1}{n-1} \cdot \frac{n-3}{n-1} \cdot \frac{1}{n-1}$. Thus,

$$\begin{aligned} E(X_i X_j) &= \left(\frac{1}{n-1}\right)^2 + \frac{n-2}{n-1} \cdot \frac{1}{n-1} \cdot \frac{n-3}{n-1} \cdot \frac{1}{n-1} \\ &= \left(\frac{1}{n-1}\right)^2 \left(\frac{(n-1)^2 + (n-2)(n-3)}{(n-1)^2}\right). \end{aligned}$$

Hence,

$$\begin{aligned} \mathrm{Cov}(X_i, X_j) &= \left(\frac{1}{n-1}\right)^2 \left(\frac{(n-1)^2 + (n-2)(n-3)}{(n-1)^2}\right) - \left(\frac{1}{n-1}\right)^2 \\ &= \frac{(n-2)(n-3)}{(n-1)^4}. \end{aligned}$$

Finally, using the fact that X_i's are identically distributed, we get

$$\begin{aligned} \mathrm{Var}(Y_n) &= \frac{n}{4}\,\mathrm{Var}(X_i) + 2\frac{\binom{n}{2}}{4}\,\mathrm{Cov}(X_i, X_j) \\ &= \frac{n}{4}\frac{n-2}{(n-1)^2} + \frac{n(n-1)}{4}\frac{(n-2)(n-3)}{(n-1)^4} \\ &= \frac{n(n-2)}{4(n-1)^3}[(n-1) + (n-3)] \\ &= \frac{n(n-2)^2}{2(n-1)^3}. \end{aligned}$$

Connections to other mathematical structures and applications A round of n player Look Up and Scream can be modeled by various mathematical structures, with a pair of screams corresponding to properties of these structures. Thus, the probability distribution of the number of pairs of screams corresponds to the probability distribution of properties in these structures.

First, recall Balas [**1**] presents a way to represent the game with a function. In particular, there is a one-to-one correspondence between rounds of n-player Look Up and Scream and functions on the integers $\{1, 2, \ldots, n\}$ with no fixed points. Each player who screams in a round of the Look Up and Scream Game corresponds to a periodic point of the function with least period 2.

Second, as mentioned earlier, Bhatia [**2**] presents a way to represent a round of the game with a digraph. In particular, there is a one-to-one correspondence between rounds of n-player Look Up and Scream and digraphs with n vertices, each with out-degree 1. Each player who screams in a round of Loop Up and Scream corresponds to a vertex v such that there is a path of length 2 starting and ending at v.

Finally, there is a one-to-one correspondence between rounds of n-player Look Up and Scream and 0-1 matrices with exactly one 1 in each row and no 1's on the main diagonal (namely the adjacency matrices of the digraphs described above). Each player who screams in a round of Look Up and Scream corresponds to a 1 on the main diagonal of the square of this matrix.

Bhatia [**2**] describes two applications of Look Up and Scream. The first involves directional antennas in a wireless network. Using a certain protocol, the antennas correspond to players in Look Up and Scream, with each player who screams corresponding to an antenna that can communicate with another antenna. The second involves peers in a specially designed peer-to-peer system in which resources are shared between two peers only if each peer has a resource that the other peer wants. A pair of players who scream corresponds to an exchange of resources between two peers.

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Summary In the game Look Up and Scream, players stand in a circle, close their eyes, and on the count of three, open their eyes, with each player looking directly at another player. If two players look directly at each other, they scream and are out of the game. In this paper, the author derives a formula for the probability that there are y pairs of yells when n people play a round of the game. Using this formula, the author derives formulas for the mean and variance of the number of pairs of yells and demonstrates how to calculate the mean rounds a game will last when starting with n players. The author also presents alternative derivations for the mean and variance of the number of pairs of yells.

Solving the Noneuclidean Uniform Circular Motion Problem by Newton's Impact Method

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Sir Isaac Newton used a polygonal approximation method to show that the magnitude of the centripetal force that a particle experiences when uniformly revolving around a circle is

$$-\frac{mv^2}{r}, \tag{1}$$

where m is the particle's mass, v its uniform velocity, and r the circle's radius. In this note we use the same polygonal approximation method to prove that in noneuclidean

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(hyperbolic) geometry the magnitude of the centripetal force for a particle revolving uniformly around a circle of radius r on a hyperbolic plane of curvature -1 is

$$-\frac{mv^2}{\tanh r}. \tag{2}$$

In an earlier work [**3**], we used Newton's dynamic argument to prove (2). In that proof we used Galileo's basic law of falling bodies. In this paper we prove (2) simply by paralleling Newton's proof [**2**, p. 47] of (1), in current mathematical notation. Newton's proof consists of replacing the circular path by a n-sided regular polygon as shown in FIGURE 1(a). As the particle travels along this polygonal path, it repeatedly collides elastically with the circle, resulting in a impulsive force acting on the particle, a force directed towards the center of the circle. Newton computes the sum of these forces and then he lets the number of sides of the polygon increase to get (1). It is to be noted that all the geometric theorems, except for the *Law of Cosines*, that are employed in deriving (1) come from *absolute geometry*, the set of theorems that follow from Euclid's postulates other than the parallel postulate.

We want to show that the noneuclidean uniform circular motion problem can be solved with the same methods that Newton used to solve the same problem in Euclidean space.

In our proof of (2), we assume, since we cannot conduct noneuclidean experiments in Euclidean space, that Newtonian mechanics holds in any infinitesimal region of an noneuclidean plane. This is a reasonable assumption to make because an infinitesimal neighborhood of any point on a noneuclidean plane is euclidean [**1**, pp. 111 and 152]. It follows from this assumption, that we can prove (2) using the same physical arguments that Newton used to prove (1).

The geometry Before proving (2), we give some geometric results needed in our proof. Noneuclidean and Euclidean geometries share many theorems and constructions —everything except those that depend on the parallel postulate. The common ground among the geometries is called *absolute geometry*. For example, the theorem that a tangent to a circle is perpendicular to the circle's radius at the point of contact is a theorem of absolute geometry. So too is the side-angle-side congruency theorem. An example of a common construction is inscribing various regular polygons within a circle. These two theorems and this construction from absolute geometry, which are used to prove (1), can be used to prove (2).

The Law of Cosines can be used to prove (1), but that law depends on the parallel postulate. In hyperbolic geometry the Law of Cosines takes the form [**1**, pp. 102–104]

$$\cosh(BC) = \cosh(AB)\cosh(AC) - \sinh(AB)\sinh(AC)\cos\alpha,$$

where BC, AB, and AC are the lengths of the sides of the triangle ABC and $\alpha = \angle BAC$, which is different from the Euclidean form.

In our proof of (2), we use this hyperbolic law, which is the only difference between our proof and Newton's.

Proof by impact. To begin the proof, we first replace the circular path by an n-sided regular polygon path as shown in FIGURE 1(a), thus replacing the continuous motion by one with n collisions. This polygonal path is inscribed in the fixed circle of radius r centered at S. Let BC and CD be any two adjacent sides of the inscribed polygon. Draw EF tangent to the circle at C. Since triangles SCD and SCB are congruent and $EF \perp SC$, $\angle BCF = \angle DCE$.

Why would a sequence of elastic collisions produce such a path? We base our reasoning on the assumption that Newtonian mechanics holds within the infinitesimal

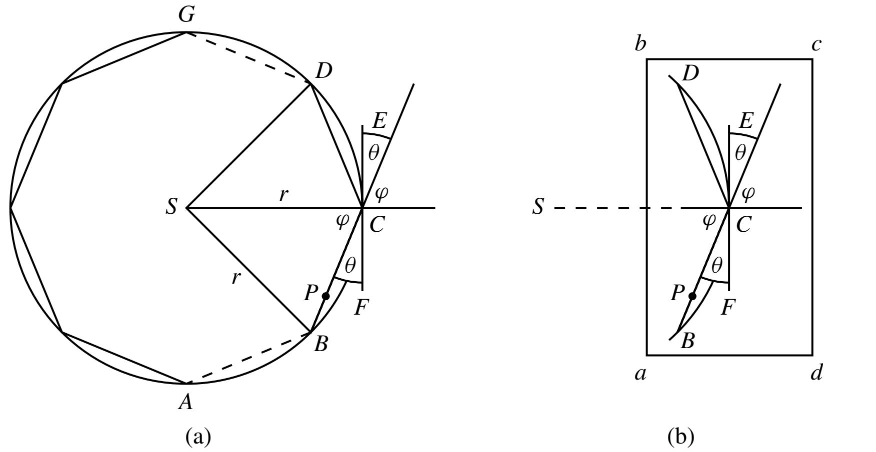

Figure 1 Particle P collides with the hyperbolic circle at C

region where the collision occurs. If the particle collides elastically with the circle at C, it is reflected off the tangent EF at C in such a way that $\angle BCF$ (incident angle) is equal to the angle of reflection and it has a rebound speed of v. But $\angle BCF = \angle DCE$, thus making $\angle DCE$ the reflection angle. Therefore, the particle will be reflected along CD with speed v. The same argument equally applies to the other collision points. This shows that the particle will continue to travel along the regular polygon's perimeter of FIGURE 1(a) with constant speed v after each collision.

Next we compute the force that the particle experiences when colliding with the circle at C. Again recall that Newtonian mechanics is assumed to hold in any infinitesimal region about point C (FIGURE 1(b)). Before colliding with the circle, the particle's linear momentum, relative to the rectangular axes CS and CE, is $\vec{M}_{BC} = (mv\sin\theta, mv\cos\theta)$, and after the collision it becomes $\vec{M}_{CD} = (-mv\sin\theta, mv\cos\theta)$ where $\theta = \angle BCF$. Therefore, the particle's total change of momentum, denoted by $\Delta\vec{\wp}$, is given by $\Delta\vec{\wp} = \vec{M}_{CD} - \vec{M}_{BC} = (-2mv\sin\theta, 0)$. Note that $\Delta\vec{\wp}$ points inward along the radius. Now let $\vec{F}$ be the force that the particle experiences while colliding with the circle, a force that acts for a very short time duration Δt. By Newton's second law of motion, we have $\Delta\vec{\wp} = \vec{F}\Delta t$, and so $\vec{F}\Delta t = (-2mv\sin\theta, 0)$. The name *impulse* is given to the product $\vec{F}\Delta t$ and its magnitude is given by

$$f\,\Delta t = -2mv\sin\theta, \tag{3}$$

where f denotes the magnitude of the force $\vec{F}$. It follows from the direction of $\Delta\vec{\wp}$ that the direction of the force f is toward the circle's center S along the radius SC.

Let $\Delta s = BC$ and $\varphi = \angle SCB$. Then applying the hyperbolic Law of Cosines to triangle SCB,

$$\cosh r = \cosh r\cosh(\Delta s) - \sinh r\sinh(\Delta s)\cos\varphi. \tag{4}$$

Since $SC \perp EF$, we have $\varphi = \frac{\pi}{2} - \theta$. Therefore, (4) can be rewritten, with the help of the half-angle formulas for hyperbolic trigonometric functions, as

$$\sin\theta = \frac{\sinh\frac{\Delta s}{2}}{\tanh r\cosh\frac{\Delta s}{2}}.$$

Next, substituting the last equation into (3) and rewriting, we find

$$f\,\Delta t = -m\frac{v\Delta s}{\tanh r}\left(\frac{\frac{\sinh\frac{\Delta s}{2}}{\frac{\Delta s}{2}}}{\cosh\frac{\Delta s}{2}}\right), \tag{5}$$

as the magnitude of the impulse at C, which is the same at the other collision points. Since there are n collision points, summing all the corresponding forces gives

$$f n\Delta t = -m\frac{v(n\Delta s)}{\tanh r}\left(\frac{\frac{\sinh\frac{\Delta s}{2}}{\frac{\Delta s}{2}}}{\cosh\frac{\Delta s}{2}}\right). \tag{6}$$

Next, let $n \to \infty$, so that the number of sides of the polygon increases without bound. Then $\Delta s \to 0$, $n\Delta s \to L$, and $n\Delta t \to T$, where L and T are the length of the circumference and the time the particle takes to travel around the circle, respectively. Thus, since $\lim_{x\to 0}(\sinh x)/x = 1$ (which easily follows from L'Hospital's rule), (6) becomes

$$fT = -m\frac{v}{\tanh r}L,$$

or, since $L = vT$,

$$f = -m\frac{v^2}{\tanh r},$$

the desired centripetal force. This completes the proof. ■

It is curious to note that in the hyperbolic plane, as the radius of a circular path becomes larger and larger, the limiting magnitude of force is mv^2, quite in contrast with the Newtonian case where the force dies out as the path of motion approaches a straight line. Perhaps this is not too surprising. If we fix a point on the circle and move the center farther and farther away along a line, the limiting shape is not a line, but a special curve called a *horocycle*; since this curve is not straight, force is required to keep the particle moving along it.

Elliptic geometry In the above proof, we can simply replace the hyperbolic Law of Cosines by the elliptic one to prove that the centripetal force in elliptic geometry (for a sphere of radius 1) is

$$-\frac{mv^2}{\tan r}.$$

In this case, when r approaches $\pi/2$, the force becomes 0, which is fitting since then the circular motion is along a great circle.

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Summary We compute the centripetal force exerted on a particle moving uniformly on the circumference of a noneuclidean circle using Newton's impact method.

Sums of Evenly Spaced Binomial Coefficients

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Two of the first identities encountered in a discrete mathematics course are the following finite sums of binomial coefficients. For $n \geq 0$,

$$\sum_{k\geq 0} \binom{n}{k} = 2^n \tag{1}$$

and for $n \geq 1$,

$$\sum_{k\geq 0} \binom{n}{2k} = 2^{n-1}. \tag{2}$$

The sums are finite since $\binom{n}{k} = 0$ when $k > n$. Both of these identities have elementary combinatorial proofs. But when $r \geq 3$, the sum $\sum_{k\geq 0} \binom{n}{rk}$ is rarely mentioned because its closed form is more *complex*. (See Gould [**1**]. A special case appears in [**3**] as problem 1.42(f).)

THEOREM 1. *For $n \geq 0$ and $r \geq 1$,*

$$\sum_{k\geq 0} \binom{n}{rk} = \frac{1}{r} \sum_{j=0}^{r-1} (1+\omega^j)^n, \tag{3}$$

where $\omega = e^{i2\pi/r}$ is a primitive rth root of unity.

Notice that when $r = 1$ or 2, we have $\omega = 1$ or -1, respectively, and the formulas in equations (1) and (2) are directly obtained. When $r = 3$, we have

$$\omega = e^{i2\pi/3} = \frac{-1+\sqrt{3}\,i}{2}$$

and then Theorem 1 yields, for $n \geq 0$,

$$\sum_{k\geq 0} \binom{n}{3k} = \frac{2^n + m}{3},$$

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where m depends on n and is equal to 2, 1, −1, −2, −1, 1, when n is congruent to 0, 1, 2, 3, 4, 5 (mod 6), respectively. Likewise when $r = 4$, we have $\omega = i$, and we get

$$\sum_{k\geq 0}\binom{n}{4k} = \frac{2^n + m2^{\lceil n/2\rceil}}{4},$$

where $m = 2, 1, 0, -1, -2, -1, 0, 1$, when $n \geq 1$ is congruent, respectively, to 0, 1, 2, 3, 4, 5, 6, 7 (mod 8). (When $n = 0$, this formula needs to be adjusted, since $0^0 = 1$.)

A generalization of Theorem 1 (which appears in Gould [**1**] in modified form) also has an attractive closed form.

THEOREM 2. *For any integers* $0 \leq a < r$ *and* $n \geq 0$*,*

$$\sum_{k\geq 0}\binom{n}{a+rk} = \frac{1}{r}\sum_{j=0}^{r-1}\omega^{-ja}(1+\omega^j)^n, \tag{4}$$

where $\omega = e^{i2\pi/r}$ *is a primitive* r*th root of unity.*

While Theorems 1 and 2 have succinct algebraic explanations using the binomial theorem (see [**2**], [**3**]), our goal is to prove them combinatorially. In a combinatorial proof, an identity is proved by counting a problem in two different ways. Our proofs will utilize the graph C_r, the *directed, looped cycle graph* with vertex set $V = \{0, 1, \ldots, r-1\}$ such that for each vertex j, there is an arc to vertex j and $j + 1$ (mod r). (See Figure 1.) We define an n-walk to be a walk on C_r that takes exactly n steps. A walk that begins and ends at the same vertex is said to be *closed*; otherwise it is *open.* For example, when $r = 5$, the walk 3, 4, 4, 0, 1, 1, 1, 2 is an open 7-walk. It makes 4 forward moves and 3 stationary moves. Another way to describe this walk would be

$$X = (x_0; x_1, x_2, x_3, x_4, x_5, x_6, x_7) = (3; F, S, F, F, S, S, F)$$

where x_0 indicates the initial vertex and the other values of x_i indicate whether the ith step is forward or stationary. Clearly, an n-walk that begins at x_0 and makes m forward moves will end up at vertex $x_0 + m$ (mod r).

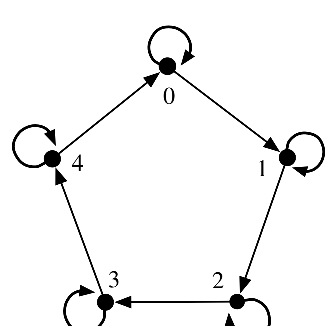

Figure 1 The looped cycle graph C_5.

Combinatorial Proof of Theorem 1

QUESTION. How many closed n-walks on C_r begin at vertex 0?

Answer 1. Starting at vertex 0, there are $\binom{n}{m}$ n-walks that take m forward steps. For a walk to be closed, m must be a multiple of r. Consequently, our first answer is simply $\sum_{k\geq 0}\binom{n}{kr}$.

Answer 2. First we observe that there are as many closed n-walks that start at vertex 0 as start at vertex 1 or 2 or ... or $r-1$. Thus it suffices to prove that the total number of closed n-walks on C_r is $\sum_{j=0}^{r-1}(1+\omega^j)^n$. We accomplish this by assigning each n-walk (whether it be open or closed) a *weight* that will depend on its initial vertex and the number of forward moves. Specifically, an n-walk with initial vertex $x_0 = j$ that makes m forward moves will be assigned a weight of ω^{jm}. The 7-walk on C_5 in the previous example has $j = 3$, and $m = 4$ and therefore has weight $\omega^{12} = \omega^2$ since $\omega^5 = 1$. Equivalently, a walk that begins at vertex j and ends at vertex $j+m \pmod r$ has weight ω^{jm}. In particular, any closed walk will have weight $\omega^0 = 1$.

Another way to think of the weight of a walk beginning at vertex j is that each stationary step is given weight 1 and each forward step in the walk is given weight ω^j, and the weight of the walk is defined as the product of the weights of its steps. Consequently, the *total* weight of *all* n-walks that begin at j is $(1+\omega^j)^n$, since each $(1+\omega^j)$ represents a choice in our walk to make a stationary or forward move. (Alternatively, $(1+\omega^j)^n = \sum_{k\geq 0}\binom{n}{k}\omega^{jk}$ is the sum of the weights of all n-walks starting at j since $\binom{n}{k}\omega^{jk}$ is the total weight of all such walks with k forward steps.) Summing over all possible starting points,

$$\sum_{j=0}^{r-1}(1+\omega^j)^n \tag{5}$$

counts the total weight of all n-walks (open and closed) on C_r.

Our goal is to show that (5) counts the total number of all *closed* n-walks on C_r. Since each closed walk has weight 1, it suffices to show that the *total weight of all open walks is zero*. Consider an open walk X_0 that begins at vertex 0 and ends at vertex $m \neq 0$. Then X_0 generates the orbit $\{X_0, X_1, \ldots, X_{r-1}\}$ where walk X_j starts at vertex j, and then follows the same forward and stationary instructions as X_0, ending at vertex $j+m \pmod r$, with weight ω^{jm}. Summing a finite geometric series, the total weight of the n-walks in this orbit is

$$\sum_{j=0}^{r-1}\omega^{jm} = \frac{1-\omega^{mr}}{1-\omega^m} = 0,$$

since $\omega^r = 1$ and $\omega^m \neq 1$. Since every open walk appears in exactly one orbit, each with total weight zero, the total weight of all open walks is zero, as desired. Summarizing, for walks on C_r,

$$\begin{aligned}\text{the number of closed } n\text{-walks} &= \text{the total weight of all closed } n\text{-walks}\\ &= \text{the total weight of all } n\text{-walks}\\ &= \sum_{j=0}^{r-1}(1+\omega^j)^n.\end{aligned}$$

Hence, the number of closed n-walks that begin at 0 is $\frac{1}{r}\sum_{j=0}^{r-1}(1+\omega^j)^n$. ■

Combinatorial Proof of Theorem 2

In this proof, an n-walk on C_r that starts at vertex j and makes m forward moves is defined to have weight $\omega^{-ja}\omega^{mj} = \omega^{(m-a)j}$. Hence any walk that makes $a + rk$ forward moves has weight $\omega^{rkj} = 1$. Just like in the proof of Theorem 1, the total weight of all n-walks on C_r is $\sum_{j=0}^{r-1} \omega^{-ja}(1 + \omega^j)^n$. The theorem follows since the walks that make forward progress $m \neq a + rk$ can be placed into orbits of total weight $\sum_{j=0}^{r-1} \omega^{(m-a)j} = 0$. ■

Theorems 1 and 2 can also be expressed in terms of trigonometric functions [**1**], sometimes without mentioning any complex numbers. Suppose $\nu = e^{i\pi/r}$ is a primitive $2r$th root of unity so that $\nu^2 = \omega$. Then using Euler's formula, $e^{-i\theta} + e^{i\theta} = 2\cos\theta$, we may write the summand as

$$(1 + \omega^j)^n = [\upsilon^j(\upsilon^{-j} + \upsilon^j)]^n = \upsilon^{nj}(e^{-i\pi j/r} + e^{i\pi j/r})^n = \upsilon^{nj}\left(2\cos\frac{\pi j}{r}\right)^n.$$

In particular, if n is a multiple of r, say $n = qr$, then

$$\sum_{k \geq 0} \binom{n}{rk} = \frac{2^n}{r} \sum_{j=0}^{r-1} (-1)^{qj} \left(\cos\frac{\pi j}{r}\right)^n \tag{6}$$

can be expressed entirely with real numbers. This is the form presented in [**1**]. Likewise, Theorem 2 simplifies to the same right hand side of (6) when $n = qr + 2a$.

Where do we go from here? A natural problem might be to try to count walks on other graphs to discover other identities. Conversely, we hope this technique may allow us to combinatorially understand other identities that mix binomial coefficients with complex numbers. For example, Identity 2.24 in [**1**] says for $r > 1$,

$$\sum_{k \geq 1} \frac{1}{\binom{kr}{r}} = \sum_{k=1}^{r-1} -\omega^k(1 - \omega^k)^{r-1} \log\frac{1 - \omega^k}{-\omega^k},$$

where ω is a primitive rth root of unity. Perhaps with the right combinatorial perspective, this identity will not appear nearly so complex after all.

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Summary We provide a combinatorial proof of a formula for the sum of evenly spaced binomial coefficients, $\sum_{k\geq 0}\binom{n}{rk}$. This identity, along with a generalization, are proved by counting weighted walks on a graph.

How Long Until a Random Sequence Decreases?

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Waiting for the fall Imagine observing a stream of random real numbers: If you saw the sequence

$$0.0478, 0.1429, 0.1667, 0.2204, 0.8124, 0.8226, 0.3101 \dots,$$

with the first decrease occurring in the seventh position, you might feel that this was an unusually long time to wait for that first decrease—even if you're not exactly sure how long such a run "usually" lasts in a random sequence. The average position of the first decrease in a stream of random numbers depends on precisely what you mean by random; that is, it depends on the distribution of random numbers that you're sampling. Surprisingly, though, for *continuous* distributions, the question has a very specific (and delightful) answer that is independent of how the random numbers are distributed.

We will uncover the answer in due time—in Proposition 2, to be precise. In the meantime, exercise your intuition by making a guess in advance about how long, on average, a monotone run like the one above will last in a sequence of random numbers. If you've been around math much, you can probably make a shrewd guess based solely on the fact that I described the answer as "delightful." But we will start by discussing a special case of the problem, when the random numbers are generated simply by rolling dice.

The answers to these questions have been known to specialists for some time. In fact, one can find the answer to the main question as an exercise in Knuth [**5**], and most of the results here can be found, in a more general setting, in Guy Louchard's thorough analysis of monotone runs [**6**]. However, they do not seem to be well known generally, despite their accessibility and interest for students with a basic undergraduate calculus background.

The die-rolling game One of the fringe benefits of teaching a course on probability and statistics is that it affords an excellent excuse to keep an assortment of toys on my desk, especially all sorts of dice. This article had its beginning when I was rolling an ordinary 6-sided die and got what I felt was an unusually long run before the first decrease in the numbers occurred. It might have been something like this:

$$1, 3, 3, 3, 4, 6, 5$$

and I decided to make a game of it: I would award myself 7 points for that run, since I got to roll seven times (including the final, decreasing roll that ended the game). Naturally, I was wondering what a typical score in my new game might be. But I also had my eye on the 4- and 20-sided dice lying nearby, and wondered if one of those might give me a better chance to get a large score. (Exercise your intuition again: can I expect to get a better score by choosing one of the other dice?)

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Mathematically, an n-sided die is modeled by a discrete random variable that is uniformly distributed over the set $\{1, 2, \dots, n\}$; by this we mean that all of the outcomes in the set occur with the same probability, $1/n$. But we're not interested in single rolls of the die; rather, we want to study nondecreasing runs from repeated rolls of the die, and that motivates the following. Given any random variable X we define an associated random variable $R(X)$, the *run-length variable* for X, as follows: We sample X until the first decrease occurs, then let $R(X)$ be the total number of samples we took, including the final decrease that ends the experiment. Call X the *underlying variable* of $R(X)$.

The $R(X)$ notation emphasizes the fact that the experiment depends on the underlying variable X, but we'll suppress the argument and simply refer to R when the underlying variable is clear from context. And we'll call the run-length variable R_n when X is an n-sided die roll.

Our goal now is to study the expected value of R, particularly when the underlying variable is an n-sided die roll. Informally, the expected value of a random variable is the long-term average of its outcomes; by definition, the expected value of R is

$$E[R] = \sum_r r f(r),$$

where the summation is over all possible outcomes r that might occur, and $f(r)$ denotes the probability of getting outcome r.

No matter what the underlying variable is, we always get an outcome of at least 2 for R. On the other hand, there's no upper bound on the potential length of a nondecreasing run, so the possible outcomes of R are $\{2, 3, 4, \dots\}$, and we can rewrite the expected value calculation with more explicit limits of summation:

$$E[R] = \sum_{r=2}^{\infty} r f(r).$$

Next we find an explicit formula for $f(r)$ in the case of an n-sided die roll. Take $n \geq 2$, and let $a_n(r)$ denote the number of nondecreasing sequences of length r that can be formed from the set $\{1, 2, \dots, n\}$. This is a problem of selection with repetition allowed, and any combinatorics text will tell you that $a_n(r)$ is given by a binomial coefficient: $a_n(r) = \binom{n+r-1}{n-1}$. In fact, we do not need to defer to a text for this: Imagine making your nondecreasing selection by distributing r stones among boxes numbered 1 through n, all in a row. Place a stick between each adjacent pair of boxes; now, let the boxes vanish (leaving their contents behind). What remains is a sequence of r stones and $(n-1)$ sticks which uniquely codes for your selection, and the number of such sequences is counted by the binomial coefficient we have given.

With this notation the probability that a sequence of length r is nondecreasing is $a_n(r)/n^r$. The probability that a random sequence of length r decreases for the first time in the last position, then, is the probability that it increases for $r-1$ steps, minus the probability that it increases for r steps:

$$f_n(r) = \frac{a_n(r-1)}{n^{r-1}} - \frac{a_n(r)}{n^r}, \tag{1}$$

which can simplified (just combine fractions and cancel factorials) to

$$f_n(r) = \binom{n+r-2}{r} \cdot \frac{(r-1)}{n^r}. \tag{2}$$

With a formula for the mass function established, we can proceed to the expected value problem.

PROPOSITION 1. *The expected value of R_n is given exactly by*

$$E[R_n] = \left(\frac{n}{n-1}\right)^n$$

Proof. With formula (2) in hand, we can write the summation for the expected value as

$$\sum_{r=2}^{\infty} r \cdot \binom{n+r-2}{r} \cdot \frac{(r-1)}{n^r} \qquad (*)$$

and massage the form until the sum can be evaluated, as follows:

$$\begin{aligned}(*) &= n(n-1)\sum_{r=2}^{\infty}\binom{n+r-2}{r-2}\left(\frac{1}{n}\right)^r \\ &= \frac{n-1}{n}\sum_{r=0}^{\infty}\binom{n+r}{r}\left(\frac{1}{n}\right)^r \\ &= \frac{n-1}{n}\left(1-\frac{1}{n}\right)^{-(n+1)},\end{aligned}$$

the last line following from the binomial series expansion of $\left(1-\frac{1}{n}\right)^{-(n+1)}$ which appears in the previous step. The series does converge, since we are assuming $n \geq 2$, and of course the resulting expression reduces to the form in the statement of the proposition. ∎

Let's look back at a few questions we can now answer about the die-rolling game:

1. By formula (2), the probability that I would get a score of 7, using an ordinary 6-sided die as in the example, is just

$$f(7) = \frac{\binom{11}{7}\cdot 6}{6^7},$$

which is about 0.007—small enough that you might suspect my example is fictitious. (With a little more work you can check that the probability that I would get a score of 7 *or more* is just a tiny bit less than 1%, which is probably a more relevant fact.)

2. By Proposition 1, the average score for a game with a 6-sided die would be $(6/5)^6$, or just barely under 3.
3. Since the formula $(n/(n-1))^n$ is strictly decreasing in n (a popular exercise), I'd have a higher score, on average, if I switched to a 4-sided die, and a lower score if I used the 20-sided die. In fact, to maximize your score, your best bet for this game would be to toss a coin with sides labelled 1 and 2. You'd expect an average score of 4 in that case.

If you're still thinking about the question posed in the introduction, you might stop to consider this: What can you say about the expected value of R if X has a very large number of equally likely outcomes?

A variation: strictly increasing die rolls If we change the rules of the die-rolling game slightly to insist on strictly increasing numbers, we get slightly different results for the mass function and expected score. In the strictly increasing game, for example, rolling

$$1, 3, 5, 5$$

would cause the game to end with a score of 4. We can briefly establish results analogous to those of the previous section for the strictly increasing game. The expected value calculation may seem even simpler, as it uses the more familiar version of the binomial theorem, where the exponent is a positive integer.

Given any random variable X, define another random variable $R_s(X)$ as follows: we sample X until we get a result which is *not* strictly greater than the previous result, then let $R_s(X)$ be the total number of samples we took. If X represents an n-sided die roll, then the probability mass function for $R_s(X)$ is given by

$$f_s(r) = \binom{n+1}{r} \cdot \frac{(r-1)}{n^r}. \tag{3}$$

The details are left as an exercise; the derivation is very similar to the nondecreasing case. And if X represents an n-sided die roll then the expected value of $R_s(X)$ is given by

$$E[R_s(X)] = \left(\frac{n+1}{n}\right)^n \tag{4}$$

To verify this, notice that in this variant we have an upper bound on the possible scores: If we use an n-sided die, then our score must come from the set $\{2, \dots, n+1\}$. That means that the expected value calculation involves only a finite sum:

$$\begin{aligned} E[R_s] &= \sum_{r=2}^{n+1} r \cdot \binom{n+1}{r} \frac{(r-1)}{n^r} \\ &= \left(\frac{n+1}{n}\right) \sum_{r=0}^{n-1} \binom{n-1}{r} \left(\frac{1}{n}\right)^r \\ &= \left(\frac{n+1}{n}\right) \left(1 + \frac{1}{n}\right)^{n-1}, \end{aligned}$$

with the last line following by the binomial theorem. And this simplifies to $\left(\frac{n+1}{n}\right)^n$ as claimed.

QUESTION. *In the strictly increasing game, what sort of die (how many sides) should you choose to maximize your expected score?*

The continuous game Now, instead of rolling dice to generate our random numbers, suppose we have a continuous (real) random variable X as our source of randomness.

The only assumption we will make is that X is described by a probability density function—that is, there is a nonnegative function $p(x)$ on $\mathbb{R}$ with the property that the probability that X is between a and b is given by

$$P(a \le X \le b) = \int_a^b p(x)\,dx.$$

With a continuous random variable, there is zero probability of any sample duplicating an earlier number in the sequence. Therefore, the probability that a randomly generated sequence of length r is nondecreasing is the same as the probability that it is strictly increasing, $1/r!$ in both cases. As in equation (1), we can compute the probability of a monotone run of length as a difference of probabilities:

$$f(r) = \frac{1}{(r-1)!} - \frac{1}{r!} = \frac{(r-1)}{r!} \tag{5}$$

PROPOSITION 2. *Let R be the run-length variable for any continuous random variable X. Then the expected value of R is exactly*

$$E[R] = e,$$

the base of the natural logarithm.

Proof. This is just an easy corollary of the formula for the mass function; working directly from the definition of expected value we have

$$E[R] = \sum_{r=2}^{\infty} r f(r) = \sum_{r=2}^{\infty} \frac{r(r-1)}{r!} = \sum_{r=2}^{\infty} \frac{1}{(r-2)!}$$

and this last expression is exactly the beloved series expansion

$$\frac{1}{0!} + \frac{1}{1!} + \frac{1}{2!} + \frac{1}{3!} + \cdots = e. \qquad \blacksquare$$

Based on the previous sections, we might have arrived at this result heuristically as follows: a continuous distribution is, loosely speaking, like a distribution with infinitely many equally likely outcomes. Since the expected run length when there are n equally likely outcomes is $E[R_n] = (n/(n-1))^n$, we could have guessed that

$$E[R] = \lim_{n\to\infty} \left(\frac{n}{n-1}\right)^n$$

and this is another famous limit expression for e. (We could just as well have used a limit of formula (4) from the strictly increasing game.) There is a pleasant symmetry in the way the strictly increasing and nondecreasing versions of the discrete game approach the continuous game, matched by the use of the binomial theorem with positive and negative exponents. This can be seen in the mass functions as well as the expected value result: taking the limit as n goes to infinity in equations (2) or (3) gives the mass function for the continuous case.

The appearance of e in this problem is reminiscent of its appearance in the "Hat-Check Problem" [**2, 3**], where $1/e$ occurs as the approximate probability that a random permutation of an n-element set has no fixed points; the probabilities converge to e as n gets large. In the hat-check problem, $1/e$ is an excellent approximation to the true probability even for relatively small n. In our problem, the expected value for R_n converges much more slowly to e; roughly, you have to use an $n = 10^k$-sided die for $E[R_n]$ to match k decimal digits of e. The slow convergence of this sequence is discussed in an interesting article by Knox and Brothers [**4**].

A combinatorial connection The discrete distribution described by equation (5) can be viewed as a limiting case of either of the two families of distributions described by equations (2) and (3). Neither the families nor the limiting distribution appear to be familiar enough to have a widely-known name attached to them. For the statistically-inclined and curious, further investigation of these distributions might begin with their variance and higher moments. In the case of the run-length distribution for continuous variables, that would entail considering sums of the form

$$\sum_{r=2}^{\infty} \frac{r^k(r-1)}{r!} \tag{6}$$

for different exponents k. Starting with $k = 1$, this will yield a sequence beginning

$$e, 3e, 10e, 37e, 151e, 674e, \ldots.$$

The sequence of coefficients (sequence A005493 in Sloane's index [**8**]) has a combinatorial interpretation in its own right, but may be better recognized as first differences in the sequence of Bell numbers,

$$\{B_k\} = 1, 2, 5, 15, 52, 203, 877, \ldots,$$

which count the number of ways to partition a k-element set. The connection can be seen by splitting (6) as

$$\sum_{r=2}^{\infty} \frac{r^k}{(r-1)!} - \sum_{r=2}^{\infty} \frac{r^{k-1}}{(r-1)!}$$

and recognizing these as Dobinski's summations [**1, 7**] for B_{k+1} and B_k (albeit with their first terms deleted). This fact probably doesn't afford us any winning insight into dice games, but the emergence of the Bell numbers here, hand in hand with Euler's e, seems to mark this problem as perfectly poised on the boundary between continuous and discrete mathematics, and a satisfying demonstration of the interplay between them.

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Summary Increasing runs of numbers are a naturally attractive feature in any randomly-generated sequence. Surprisingly, the average length of such runs is easy to compute and does not depend on the distribution of the random numbers, at least in the case of continuous random variables. We prove this, along with similar results for runs in sequences generated by rolling dice.

Monotonicity of Sequences Approximating e^x

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It is well known that $E_{x,\alpha}(n) = (1+\frac{x}{n})^{n+\alpha} = (1+\frac{x}{n})^n(1+\frac{x}{n})^\alpha$ converges to e^x for every fixed $x, \alpha \in \mathbb{R}$. For the two most famous sequences of this form, $\{(1+\frac{1}{n})^n\}$ and $\{(1+\frac{1}{n})^{n+1}\}$, convergence to e is strictly monotonic—consecutive terms from the first sequence strictly increase for all $n \geq 1$ while those from the second sequence strictly decrease for all $n \geq 1$. Recent notes in this MAGAZINE [**3, 4**] demonstrate these properties using little more than the arithmetic-geometric-mean inequality. Behavior of the sequences $\{E_{x,\alpha}(n)\}$ for all values of x and α is more complicated, as we show in this note.

My interest in the topic was awakened one day while playing with a calculator. I tried to investigate monotonicity experimentally by comparing $(1+\frac{x}{n})^{n+\alpha}$ with $(1+\frac{x}{n+1})^{n+1+\alpha}$. The comparisons seemed more striking when I rewrote

$$\left(1+\frac{x}{n}\right)^{n+\alpha} < \left(1+\frac{x}{n+1}\right)^{n+1+\alpha} \quad \text{as} \quad \frac{(n+x)^{n+\alpha}}{n^{n+\alpha}} < \frac{(n+x+1)^{n+\alpha+1}}{(n+1)^{n+\alpha+1}}.$$

After using the calculator to verify (for small values of n) known inequalities in the cases where $x = 1, \alpha = 0$ and where $x = 1, \alpha = 1$, I looked at corresponding inequalities for other values of x and α. For example, when $x = 9$ and $\alpha = 5$, the calculator showed that terms of the sequence $\{(1+\frac{9}{n})^{n+5}\}$ decrease monotonically:

$$\frac{10^6}{1^6} > \frac{11^7}{2^7} > \frac{12^8}{3^8} > \frac{13^9}{4^9} > \frac{14^{10}}{5^{10}} > \cdots.$$

When $x = 9$ and $\alpha = 4$, initial terms of the sequence $\{(1+\frac{9}{n})^{n+4}\}$ also decreased:

$$\frac{10^5}{1^5} > \frac{11^6}{2^6} > \frac{12^7}{3^7} > \frac{13^8}{4^8} > \frac{14^9}{5^9} > \cdots > \frac{31^{26}}{22^{26}} > \frac{32^{27}}{23^{27}}.$$

But then came a surprise. The sense of the inequality reversed between the twenty-third and twenty-fourth terms:

$$\frac{32^{27}}{23^{27}} < \frac{33^{28}}{24^{28}}.$$

The reversed sense then persisted for all larger comparisons that the calculator could make. What was going on? Rewritten in the form

$$\begin{aligned}
1^5 \cdot 11^6 &< 2^6 \cdot 10^5\\
2^6 \cdot 12^7 &< 3^7 \cdot 11^6\\
3^7 \cdot 13^8 &< 4^8 \cdot 12^7\\
&\vdots\\
22^{26} \cdot 32^{27} &< 23^{27} \cdot 31^{26}\\
23^{27} \cdot 33^{28} &> 24^{28} \cdot 32^{27}
\end{aligned}$$

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the reversal is perhaps even more striking. (When $x = 11$ and $\alpha = 5$, a reader who perseveres can find a similar reversal in the sense of the inequalities, but the change occurs a few terms later.)

Such behavior prompts a question: How do all of the sequences $\{E_{x,\alpha}(n)\}$ approach their limits? The general question does not seem to have been considered for all possible real values of x and α. In the best-known cases [**2**], $\{(1 + \frac{x}{n})^n\}$ increases monotonically for all n when $x > 0$, while $\{(1 + \frac{x}{n})^{n+\alpha}\}$ decreases monotonically for all n provided $\alpha \geq x > 0$. The same behavior governs the special case analyzed recently by Khattri [**1**]. Our example, however, shows that there is another possibility. The terms of $\{E_{9,4}(n)\} = \{(1 + \frac{9}{n})^{n+4}\}$ decrease for $n = 1, \ldots, 23$ and then increase monotonically (so it turns out) towards e^9. This shows what can happen when $\alpha < x$. In fact, we shall see that a change of direction can occur only if $0 < \alpha < x/2$. This single change of direction is as bad as it ever gets—after the change, the sequence is monotonic.

Focusing on the case where $x > 0$, here are the main results.

THEOREM. *Assume $x > 0$ and α are fixed.*

(a) *The sequence $\{E_{x,\alpha}(n)\} = \{(1 + \frac{x}{n})^{n+\alpha}\}$ increases monotonically if $\alpha \leq 0$ and decreases monotonically if $\alpha \geq \frac{x}{2}$.*

(b) *If $0 < \alpha < \frac{x}{2}$, then there exists a unique positive solution $t = t_1$ of*

$$\ln\left(1 + \frac{x}{t}\right) - \frac{x(t + \alpha)}{t(t + x)} = 0.$$

The sequence $\{E_{x,\alpha}(n)\}$ decreases monotonically for $n \leq t_1$ and increases monotonically for $n \geq t_1$.

The theorem is an immediate consequence of Proposition 1, stated and proved below, which considers functions like those defining the sequences but with a real variable, t, replacing n. As $t \to +\infty$, monotonicity of $E_{x,\alpha}(t)$ is exactly as described in the theorem and t_1, when it exists, is the unique critical value. Indeed, behavior of the real-variable function for a given x and α completely determines the behavior of the corresponding sequence with one slight ambiguity—if $n_1 < t_1 < n_1 + 1$, one does not know how $E_{x,\alpha}(n_1)$ and $E_{x,\alpha}(n_1 + 1)$ compare. For instance, the value of t_1 computed for $\{E_{9,4}(n)\}$ is approximately 22.65, while the value for $\{E_{9.01,4}(n)\}$ is about 22.44. Both sequences decrease monotonically for $n \leq 22$ and increase monotonically for $n \geq 23$. The comparison of terms on either side of t_1, however, is different for the two sequences:

$$E_{9,4}(22) = (31/22)^{26} > (32/23)^{27} = E_{9,4}(23) \text{ but } E_{9.01,4}(22) < E_{9.01,4}(23).$$

Some properties of $E_{x,\alpha}(t)$ can be inferred from typical sketches, as in FIGURE 1.

One property that is not immediately apparent from the sketches is that on the right-hand side of each sketch, the curve where $\alpha = x/2$ separates the curves that decrease monotonically for all $t > 0$ from those that first decrease and then increase. Showing that $E_{x,\alpha}$ has a single critical point when $0 < \alpha < x/2$ and no critical point otherwise will resolve this issue. Taking into account the behavior as $t \to 0^+$, the lack of critical points for other values of α will imply that curves with $\alpha \geq x/2$ monotonically decrease and those with $\alpha \leq 0$ monotonically increase for all $t > 0$. We therefore wish to prove:

PROPOSITION 1. *Fix $x > 0$ and α; then $E_{x,\alpha}(t)$, for $t > 0$, has a unique critical value, t_1, when $0 < \alpha < x/2$, and otherwise has no critical value.*

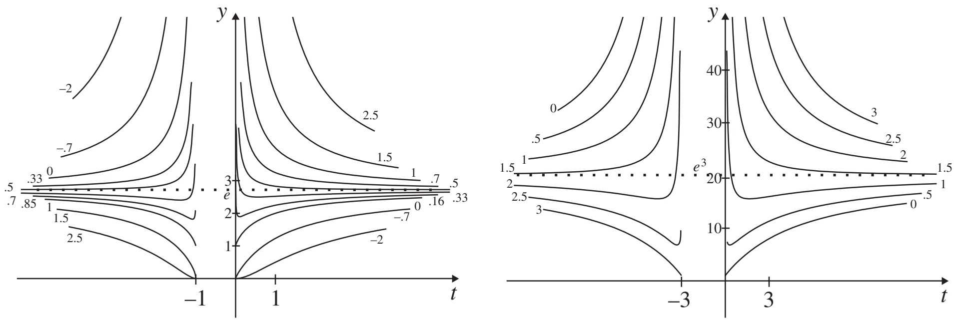

Figure 1 (a) $y = E_{1,\alpha}(t) = \left(1+\frac{1}{t}\right)^{t+\alpha}$, (b) $y = E_{3,\alpha}(t) = \left(1+\frac{3}{t}\right)^{t+\alpha}$, both for indicated values of α

Proof. Since $E_{x,\alpha}(t) > 0$ for all $t > 0$, the derivative $E'_{x,\alpha}(t) = 0$ if and only if

$$F_{x,\alpha}(t) = \frac{E'_{x,\alpha}(t)}{E_{x,\alpha}(t)} = \ln\left(1+\frac{x}{t}\right) - \frac{x(t+\alpha)}{t(t+x)} = 0.$$

For a given value of α, it therefore suffices to determine the number of positive roots of $F_{x,\alpha}(t) = 0$. Simple limit calculations show

$$\lim_{t\to 0^+} F_{x,\alpha}(t) = \begin{cases} -\infty \text{ if } \alpha > 0 \\ +\infty \text{ if } \alpha \le 0 \end{cases} \text{ and } \lim_{t\to\infty} F_{x,\alpha}(t) = 0.$$

Moreover,

$$F'_{x,\alpha}(t) = \frac{(-x^2+2\alpha x)t + \alpha x^2}{t^2(t+x)^2} = 0 \text{ iff } t = \frac{\alpha x}{x-2\alpha}.$$

As x is positive, the value $t_0 = \alpha x/(x-2\alpha)$ cannot be positive if $\alpha \le 0$, so there are no positive critical values of $F_{x,\alpha}$ in this case. The asymptotic values of $F_{x,\alpha}(t)$ computed when $\alpha \le 0$ therefore force $F_{x,\alpha}(t) > 0$ for all $t > 0$. Hence $E_{x,\alpha}(t)$ has no critical values for $t > 0$ when $\alpha \le 0$.

When $\alpha > 0$, $t_0 = \alpha x/(x-2\alpha)$ is positive only when $x - 2\alpha > 0$ or in other words, when $0 < \alpha < x/2$. The corresponding critical value of $F_{x,\alpha}$ together with the asymptotic values found above force $F_{x,\alpha}(t)$ to change sign exactly once as t goes from 0 to $+\infty$. It follows that $F_{x,\alpha}(t_1) = 0$ for a single value t_1 with $0 < t_1 < t_0$. Hence $E_{x,\alpha}(t)$ has a single positive critical value, t_1, when $0 < \alpha < x/2$. On the other hand, when $0 < x/2 \le \alpha$, $F_{x,\alpha}$ has no positive critical value. This, combined with the corresponding asymptotic information, implies $F_{x,\alpha}(t) < 0$ for all $t > 0$. So in this case, $E_{x,\alpha}(t)$ has no critical value. ∎

Proposition 1 establishes the Theorem and also the claim that $E_{x,\alpha}(t)$ with $\alpha = x/2$ separates the monotone graphs in FIGURE 1 from those that decrease and then increase. These properties require $x > 0$. When $x < 0$, the situation is similar, but the details change. As t goes from $|x|$ to $+\infty$ for a fixed negative x and $\alpha \le x < 0$, $E_{x,\alpha}(t)$ decreases monotonically. When $x < \alpha < x/2 < 0$, $E_{x,\alpha}(t)$ first increases and then decreases monotonically. And when $x/2 \le \alpha$, $E_{x,\alpha}(t)$ increases monotonically. Interested readers may confirm these properties for themselves.

Far-flung reversals We should keep in mind that all of the sequences $\{E_{x,\alpha}(n)\}$ converge to their limits very slowly. For example, $\{E_{9,4}(n)\}$ and $\{E_{9.01,4}(n)\}$, considered earlier, require more than 100,000 terms to get within .1 of their limiting values. It is conceivable that a change of direction in the growth of terms of some of the sequences might also occur very far out in those sequences. This matter is not settled by the theorem, but Proposition 1 and some of the details of its proof give good tools for investigation.

We will prove that in any collection of sequences of the form $\{E_{x,\alpha}(n)\}$ with $0 < \alpha < x/2$, where either x or α is fixed, there are sequences where the change from decrease to monotonic increase occurs beyond a point specified by an arbitrarily chosen positive integer.

To establish this assertion, we first consider the case where $\alpha = 1$ is fixed and $x > 2$. Define a ratio function $\psi_{x,1}(t) = E_{x,1}(t)/e^x = e^{-x}(1 + \frac{x}{t})^{t+1}$, which approaches 1 for large t. Define a region A of the plane by $A = \{(t, \psi_{x,1}(t)) \mid 2 \le x \le 3, t > 0\}$. The region is shown in FIGURE 2.

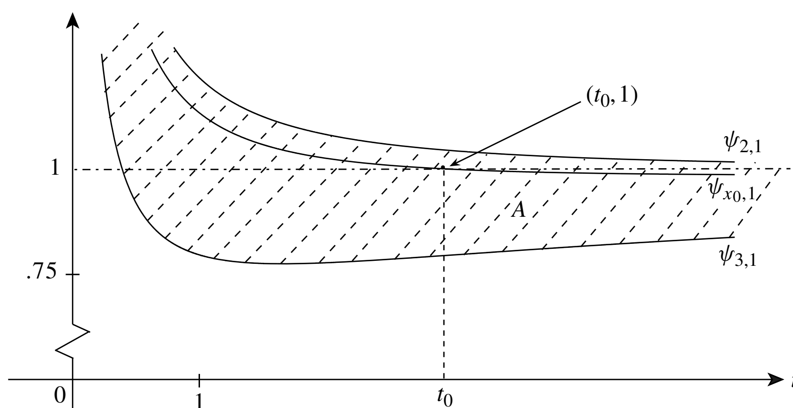

Figure 2 The region $A = \{(t, \psi_{x,1}(t)) \mid 2 \le x \le 3, t > 0\}$ where $\psi_{x,1}(t) = e^{-x}(1 + \frac{x}{t})^{t+1}$.

For any fixed value $t_0 > 0$, $\psi_{x,1}(t_0)$ is a differentiable function of x. A straightforward calculation shows that this function is strictly decreasing, which implies that it is one-to-one. If $t > 0$ varies, it follows that for all x between 2 and 3, the graphs of the curves defined by $\psi_{x,1}$ are disjoint and fill out the region A.

For any fixed value of x, the functions $\psi_{x,1}$ and $E_{x,1}$ have the same derivative with respect to t up to the factor e^{-x}. Hence they have the same critical values, and conclusions of Proposition 1 apply to $\psi_{x,1}$. Comparing outputs of the upper and lower bounding functions of A, $\psi_{2,1}(t) > 1$ for all $t > 0$ while $\psi_{3,1}(t) < 1$ for all $t \ge 1$. (These follow from Proposition 1 and the fact that $\psi_{3,1}(1) < 1$.) Let t_0 be any fixed positive value greater than 1. Since the curves $\psi_{x,1}$ fill out region A, one of them, $\psi_{x_0,1}$, must pass through the point $(t_0, 1)$, as shown in FIGURE 2. By Proposition 1, $\psi_{x_0,1}(t)$ must approach 1 through values less than 1, so its unique critical point must have t-coordinate greater than t_0. This shows that there are functions $\psi_{x,1}$ (and also $E_{x,1}$) whose critical values are arbitrarily large and sequences $\{E_{x,1}(n)\} = \{(1 + \frac{x}{n})^{n+1}\}$ that decrease for more than any specified number of terms before increasing monotonically towards their limits.

With straightforward modification, similar arguments show for any fixed $\alpha > 0$ that values of x can be found with $\alpha < x/2$ for which the functions $\psi_{x,\alpha}$ and $E_{x,\alpha}$ have arbitrarily large critical values and for which the sequences $\{E_{x,\alpha}(n)\} = \{(1 + \frac{x}{n})^{n+\alpha}\}$ decrease for a correspondingly large number of terms before increasing monotonically. When x rather than α is fixed and the collection of sequences is chosen to include all values of α with $0 < \alpha < x/2$, further modification shows that there are again functions $E_{x,\alpha}$ with arbitrarily large critical values and sequences $\{E_{x,\alpha}(n)\} = \{(1 + \frac{x}{n})^{n+\alpha}\}$

from the collection where the number of terms that decrease before monotonic increase takes hold exceeds any specified number.

As a concluding example, we compare the sequences $\{E_{2.1,1}(n)\}$ and $\{E_{2.001,1}(n)\}$, which both decrease before increasing to their limits. The minimum value of $E_{2.1,1}(t)$ occurs near $t = 14$. A calculator verifies that the first fourteen terms of the sequence $\{E_{2.1,1}(n)\} = \{(1 + \frac{2.1}{n})^{n+1}\}$ decrease while succeeding terms increase. After the sense of the inequality reverses, Proposition 1 and the Theorem guarantee that the sequence then increases monotonically towards the limiting value $e^{2.1}$. But the reversal is much less easy to spot for the sequence

$$E_{2.001,1}(n) = \frac{(n + 2.001)^{n+1}}{n^{n+1}}.$$

Write down the first five hundred or so terms for someone not in the know,

$$\frac{3.001^2}{1^2} > \frac{4.001^3}{2^3} > \frac{5.001^4}{3^4} > \frac{6.001^5}{4^5} > \cdots > \frac{502.001^{501}}{500^{501}} > \cdots,$$

and then ask: What do you think? Will this sequence always decrease, or not?

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Summary Apparently, it has not previously been observed that as $n \to \infty$, a sequence of the form $\{(1 + \frac{x}{n})^{x+\alpha}\}$ with $x > 0$ can first decrease for more than any arbitrarily specified number of terms before increasing monotonically towards the limiting value, e^x. We prove that when $0 < \alpha < \frac{x}{2}$, values for x and α can always be found so that this type of reversal in the growth of terms of the sequence is realized, and outside this range, convergence is strictly monotonic starting from the first term of the sequence.

Golden Window

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The design of the window illustrated in FIGURES 1 and 2 should please every fan of geometry. With this window in my home, whether the circular medallion of FIGURE 1 or the semi-circular arch FIGURE 2, I would offer guests a puzzle: Start with two small central circles of unit diameter, then find the radius R of the two circles on their left and right, given that a pair of congruent circles (dotted) is simultaneously tangent to all the other circles.

Math. Mag. **83** (2010) 384–390. doi:10.4169/002557010X529815. © Mathematical Association of America

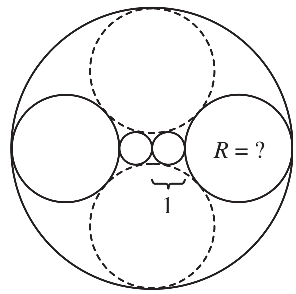


Figure 1 A puzzle

Guests could deduce, by multiple applications of the Pythagorean Theorem, for instance, that $R = \varphi \approx 1.618$, the golden ratio!

There is more: The centers of the two circles of radius R are located at distance $1 + \varphi = \varphi^2$ from the center of the window and the radius of the big circumscribing circle is the cube of the golden ratio, $1 + \varphi = \varphi^3$. Actually, the figure is replete with the golden ratio and its powers; hence the design deserves the name *golden window*.

I could spend time calling my guests' attention to the "golden" attributes of the window. To start with, it contains powers of the golden ratio from φ^0 to φ^4, as shown in FIGURE 2. It also contains various segments with *golden cuts*, as shown in the same figure below the window.

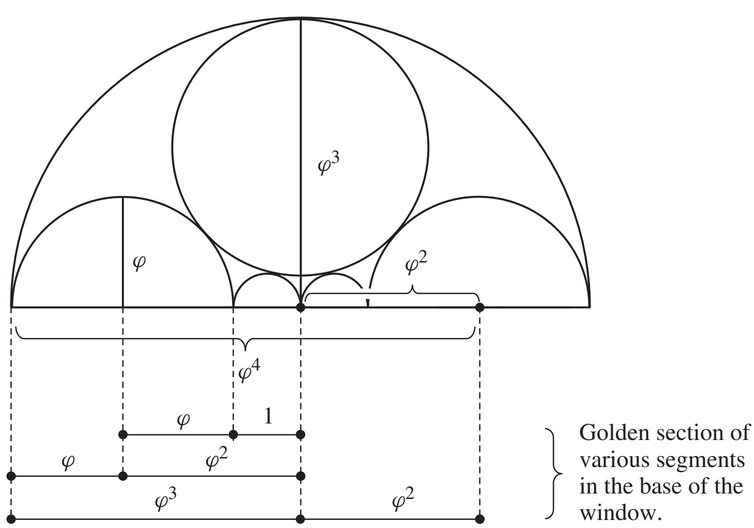


Figure 2 Golden window—proportions

Recognizing such segments is an easy game (once you establish that $R = \varphi$) if only we remember the fundamental properties of the golden ratio, namely

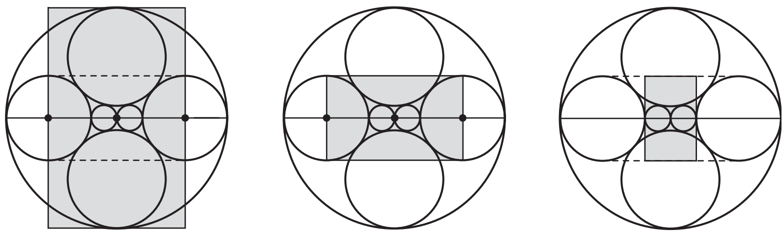
$$\varphi^n = \varphi^{n-1} + \varphi^{n-2} \quad \text{and} \quad \varphi^n = F_n\varphi + F_{n-1},$$

where F_n denotes the nth Fibonacci number, $F_1 = 1$, $F_2 = 1$, $F_3 = 2, \dots,$ with $F_{n+1} = F_n + F_{n-1}$. For small n we have:

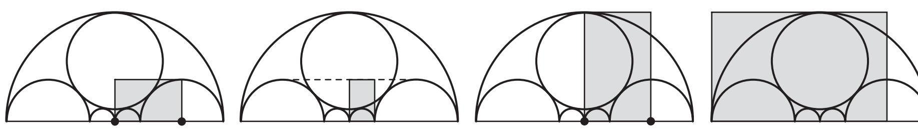
$$\varphi^2 = \varphi^1 + \varphi^0 = \varphi + 1$$
$$\varphi^3 = \varphi^2 + \varphi = 2\varphi + 1$$

$$\varphi^4 = \varphi^3 + \varphi^2 = 3\varphi + 2$$
$$\varphi^5 = \varphi^4 + \varphi^3 = 5\varphi + 3, \ldots$$

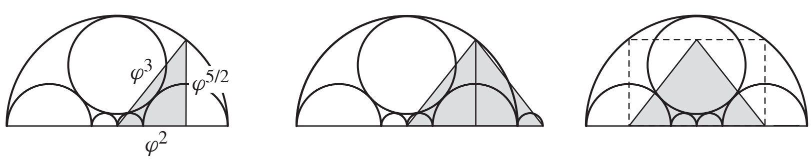
Next, I would point out various *golden rectangles* in the construction:



If the window were truncated to the upper half, I could challenge my guests to find these golden rectangles:



The culmination would be the challenge of finding the silhouette of the *Khu-fu pyramid of Giza*. Recall that the pyramid's half-silhouette makes (intentionally or not) a nearly perfect model of the so-called Kepler triangle, a right triangle whose edges form a geometric progression. The only such triangle has sides proportional to $1 : \sqrt{\varphi} : \varphi$. The shaded triangle shown below at the left has just such proportions.



Indeed, the height h can be calculated from its base φ^2 and its hypotenuse φ^3 with the Pythagorean theorem:

$$h^2 = (\varphi^3)^2 - (\varphi^2)^2 = \varphi^6 - \varphi^4 = \varphi^4(\varphi^2 - 1) = \varphi^4\varphi = \varphi^5.$$

Thus we have the triangle $(\varphi^2, \varphi^{5/2}, \varphi^3) = \varphi^2(1, \sqrt{\varphi}, \varphi)$—Kepler's golden triangle scaled by the factor φ^2. The pyramid may of course be drawn in a central position as well (the trick to see it is to apply reflective symmetry to the initial triangle).

A last challenge would be to inscribe two small circles in the upper left and right of the window. The question is: Are their centers collinear with the center of the other upper circle? And are they vertically aligned with the circles below them, or do they only seem so? The emerging rectangle (dotted lines) seems to be composed of two squares (the center of either small upper circle and the principal center would form a square's diagonal); is it indeed a square?

This would lead us to consider Descartes' circle formula [**2**], its extension [**7**], and its generalization [**5**]. But that would have to wait until after dinner.

Tools for tangent circles Readers may have solved the original puzzle—to find the radii of the circles that make the construction possible—by repeated use of the Pythagorean Theorem, but the last few questions present quite a computational challenge if this is the only tool available. A more insightful approach to circles in various

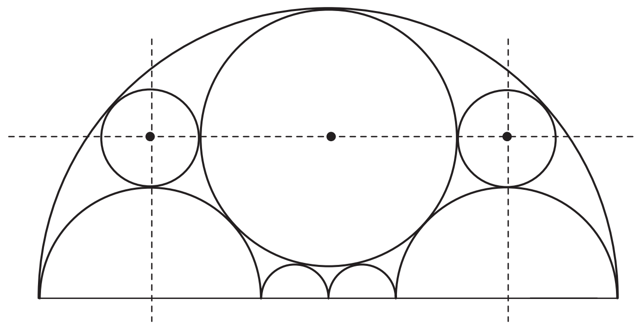


Figure 3 Are the centers aligned?

configurations starts with Descartes' theorem, its extension (which was discovered only in 2001), and finally the most general theorem. They are collected below for the convenience of the reader and as an inducement to study further the beautiful geometry of circles.

Level 1: Descartes theorem In 1643, René Descartes gave a remarkable formula that relates the radii of four mutually tangent circles [**2**]:

$$\left(\frac{1}{r_1}+\frac{1}{r_2}+\frac{1}{r_3}+\frac{1}{r_4}\right)^2 = 2\left(\frac{1}{r_1^2}+\frac{1}{r_2^2}+\frac{1}{r_3^2}+\frac{1}{r_4^2}\right). \tag{1}$$

Using the *reciprocals* of radii, that is, the curvatures, the formula reads

$$(a+b+c+d)^2 = 2(a^2+b^2+c^2+d^2), \tag{2}$$

where $a = 1/r_1$, $b = 1/r_2$, and so on. We assume that if a circle *contains* the other circles, its curvature is negative.

Descartes' formula has been rediscovered many times and its higher-dimensional generalization has also been found [**1, 10, 4**]. A system of four pairwise tangent circles is called the *Descartes configuration*, and sometimes *Soddy's circles* [**8**], after one of the re-discoverers [**10**].

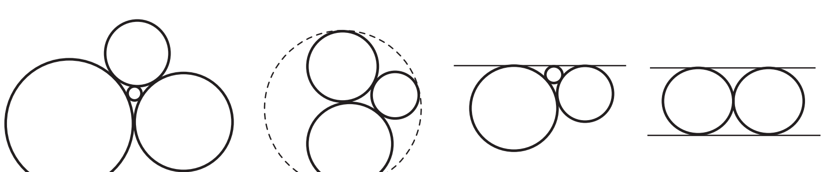


Figure 4 Examples of four circles in the Descartes configuration

One could use Descartes' formula to determine the radius of the upper corner circles in FIGURE 3. Each belongs to a Descartes configuration with three other circles with curvatures

$$a = \varphi^{-1}, \quad b = \sqrt{5}\varphi^{-3}, \quad c = -\varphi^{-3}.$$

Substituting in (2), we get $d = \sqrt{5}\varphi^{-1}$, which gives the radius $r = \varphi/\sqrt{5} = (5+\sqrt{5})/10$. This suffices to establish the collinearity of points hypothesized in the puzzle, although one does need to know the radius of the central upper circle.

Level 2: Extended Descartes theorem Note that Descartes' formula is quadratic and may be represented in matrix form. If $b_1 = 1/r_1, b_2 = 1/r_2$, etc., denote curvatures then

$$\begin{bmatrix} b_1 & b_2 & b_3 & b_4 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} b_1 \\ b_2 \\ b_3 \\ b_4 \end{bmatrix} = [0], \tag{3}$$

or—briefly—$B^T DB = 0$, with the obvious association of symbols. The *Extended Descartes* theorem was proposed by Lagarias, Mallows, and Wilks in 2002 [**7**]. In addition to the curvatures, it includes the positions of the centers (x_i, y_i), $i = 1, \ldots, 4$, and some additional variables, yet to be explained. First let us enjoy the nice matrix form:

$$\begin{bmatrix} \dot{x}_1 & \dot{x}_2 & \dot{x}_3 & \dot{x}_4 \\ \dot{y}_1 & \dot{y}_2 & \dot{y}_3 & \dot{y}_4 \\ b_1 & b_2 & b_3 & b_4 \\ \overline{b}_1 & \overline{b}_2 & \overline{b}_3 & \overline{b}_4 \end{bmatrix} \begin{bmatrix} -1 & 1 & 1 & 1 \\ 1 & -1 & 1 & 1 \\ 1 & 1 & -1 & 1 \\ 1 & 1 & 1 & -1 \end{bmatrix} \cdot \begin{bmatrix} \dot{x}_1 & \dot{y}_1 & b_1 & \overline{b}_1 \\ \dot{x}_2 & \dot{y}_2 & b_2 & \overline{b}_2 \\ \dot{x}_3 & \dot{y}_3 & b_3 & \overline{b}_3 \\ \dot{x}_4 & \dot{y}_4 & b_4 & \overline{b}_4 \end{bmatrix} = \begin{bmatrix} -4 & 0 & 0 & 0 \\ 0 & -4 & 0 & 0 \\ 0 & 0 & 0 & 8 \\ 0 & 0 & 8 & 0 \end{bmatrix}. \tag{4}$$

Note that the original Descartes formula (3) is embedded in (4). The dotted variables represent *reduced coordinates*—reduced by the corresponding radii: $\dot{x}_i = x_i/r_i$ and $\dot{y}_i = y_i/r_i$. The barred bs denote the *cocurvatures* of the circles and are defined as $\overline{b} = (\dot{x}^2 + \dot{y}^2 - 1)/b$ for each circle, but they need not concern us: For our purposes one needs only to extract from (4) three equations, $X^T DX = -4$, $Y^T DY = -4$, and $B^T DB = 0$, where X, Y, and B denote the first three columns of the third matrix, respectively.

Level 3: General circle theorem Unfortunately the crucial circles in the golden window do not form a Descartes configuration. The question is: is there a formula that would apply to not-necessarily-tangent circles? I am happy to report that there is.

Suppose you have four circles in general position (some tangent, some possibly orthogonal, etc.). Define a *circle configuration matrix* f with entries

$$f_{ij} = \frac{d_{ij}^2 - r_i^2 - r_j^2}{2r_i r_j}. \tag{5}$$

The six numbers d_{ij} denote the distances between the centers of the corresponding circles.

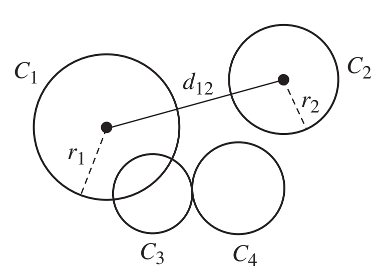


Figure 5 Four circles in general position

CIRCLE CONFIGURATION THEOREM [**6**]. *With the above notation, four circles in general position satisfy*

$$\begin{bmatrix} \dot{x}_1 & \dot{x}_2 & \dot{x}_3 & \dot{x}_3 \\ \dot{y}_1 & \dot{y}_2 & \dot{y}_3 & \dot{y}_4 \\ b_1 & b_2 & b_3 & b_4 \\ \bar{b}_1 & \bar{b}_2 & \bar{b}_3 & \bar{b}_4 \end{bmatrix} \begin{bmatrix} F_{11} & \cdots & F_{14} \\ \vdots & & \vdots \\ F_{41} & \cdots & F_{44} \end{bmatrix} \begin{bmatrix} \dot{x}_1 & \dot{y}_1 & b_1 & \bar{b}_1 \\ \dot{x}_2 & \dot{y}_2 & b_2 & \bar{b}_2 \\ \dot{x}_3 & \dot{y}_3 & b_3 & \bar{b}_3 \\ \dot{x}_4 & \dot{y}_4 & b_4 & \bar{b}_4 \end{bmatrix} = \begin{bmatrix} -1 & 0 & 0 & 0 \\ 0 & -1 & 0 & 0 \\ 0 & 0 & 0 & 2 \\ 0 & 0 & 2 & 0 \end{bmatrix}, \tag{6}$$

where F is the inverse of the configuration matrix, $F = f^{-1}$.

The truncated version for curvatures only is thus $B^T F B = 0$, or

$$\sum_{i,j} F_{ij} b_i b_j = 0 \tag{7}$$

and may be viewed as a strong generalization of the Descartes formula.

Fortunately, finding the entries of the matrix f is often quite simple and direct, without the need of equation (4). Special cases are shown in FIGURE 6, where the ijth entry is denoted as a "product of two circles," $f_{ij} = \langle C_i, C_j \rangle$, called in [**5**] the "Pedoe product," since it may indeed be traced to D. Pedoe [**9**, p. 155].

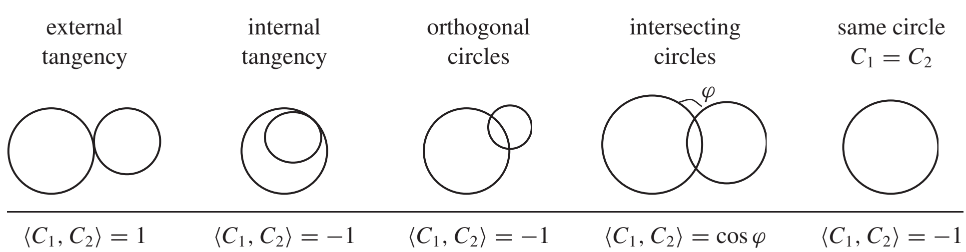


Figure 6 Pedoe inner product of two circles (possible entries of matrix f)

Note that in the special case of mutually tangent circles, FIGURE 6 shows us that the matrix f is the one in (3). Its inverse is $F = f^{-1} = (1/4)f$; thus the Descartes formula (including the extended version) follows as a very special case.

The theorem may be used to solve the puzzle. By the way, the design is a special case of a *lens chain*—a collinear system of tangent circles simultaneously tangent to two congruent disks; more on this may be found in [**6**].

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Summary Finding appearances of the golden ratio in various nooks and crannies of mathematics brings delight, often surprise. This note presents, in the form of a puzzle, a configuration of circles that is replete with the golden ratio. But that is only the surface. One tool to analyze such figures is the "master matrix equation" that rules circle (and n-sphere) configurations. This equation generalizes the famous circle theorem of Descartes (known also as Soddy's kissing circle theorem).

Questions answered The first two questions posed at the end of this note have positive answers: The centers of the little corner circles are indeed aligned with the centers of the adjacent circles. Their exact positions and radii are shown in the FIGURE 7.

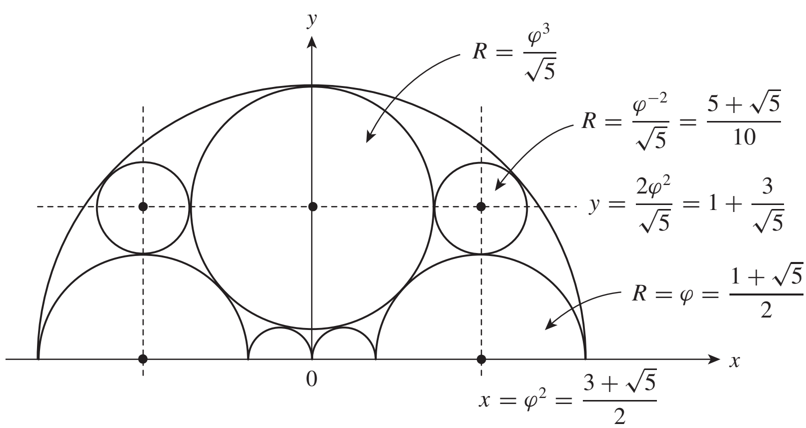

Figure 7 Some answers

As to the "square," it turns out that it is actually a rectangle of proportion 2: $\sqrt{5}$, as can be seen above.

PROBLEMS

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PROPOSALS

To be considered for publication, solutions should be received by May 1, 2011.

1856. *Proposed by H. A. ShahAli, Tehran, Iran.*

(a) Determine all the positive integers n for which there exists an $n \times n$ array of entries in $\{0, 1\}$ such that the row sums are pairwise distinct and the column sums are all equal.
(b) Determine all such positive integers n under the additional restriction that every row has at least one entry equal to 1.

1857. *Proposed by Cezar Lupu (student), University of Bucharest, Bucharest, Romania; and Tudorel Lupu, Decebal High School, Constanta, Romania.*

Let ABC be an arbitrary triangle with $a = BC$, $b = AC$, and $c = AB$. The points A_1, B_1, and C_1, on the segments BC, AC, and AB, respectively, satisfy that $AB + BA_1 = AC + CA_1$, $BC + CB_1 = BA + AB_1$, and $CA + AC_1 = CB + BC_1$. Prove that

$$\frac{\text{Area}(A_1B_1C_1)}{\text{Area}(A\,B\,C)} \leq \frac{9abc}{4(a+b+c)(a^2+b^2+c^2)}.$$

1858. *Proposed by Herman Roelants, Center for Logic, Institute of Philosophy, University of Louvain, Leuven, Belgium.*

Let $p \geq 3$ be an odd integer. Prove that the equation $u^p + 4^{p-1} = v^2$ has nonzero rational solutions (u, v), if and only if, the equation $x^p + y^p = z^p$ has nonzero integer solutions (x, y, z).

Math. Mag. **83** (2010) 391–397. doi:10.4169/002557010X529824.

We invite readers to submit problems believed to be new and appealing to students and teachers of advanced undergraduate mathematics. Proposals must, in general, be accompanied by solutions and by any bibliographical information that will assist the editors and referees. A problem submitted as a Quickie should have an unexpected, succinct solution. Submitted problems should not be under consideration for publication elsewhere.

Solutions should be written in a style appropriate for this MAGAZINE.

Solutions and new proposals should be mailed to Bernardo M. Ábrego, Problems Editor, Department of Mathematics, California State University, Northridge, 18111 Nordhoff St, Northridge, CA 91330-8313, or mailed electronically (ideally as a LaTeX or pdf file) to mathmagproblems@csun.edu. All communications, written or electronic, should include **on each page** the reader's name, full address, and an e-mail address and/or FAX number.

1859. *Proposed by Valmir Krasniqi, Department of Mathematics, University of Prishtina, Prishtinë, Republic of Kosova.*

Let α be a positive real number and f be a nonnegative function on [0, 1] such that

$$\int_x^1 (f(t))^\alpha \, dt \geq \int_x^1 t^\alpha \, dt \text{ for all } 0 \leq x \leq 1.$$

Prove that $\int_0^1 (f(t))^{\alpha+\beta} \, dt \geq \int_0^1 (f(t))^\alpha t^\beta \, dt \geq \int_0^1 t^{\alpha+\beta} \, dt$ for every positive real β.

1860. *Proposed by Marian Tetiva, National College "Gheorghe Roşca Codreanu," Bârlad, Romania.*

Let α be a complex number such that $|\alpha| > 1$ and let n be an integer such that $n > 2$. Prove that at least $n - 2$ roots of the equation $z^n + \alpha z^{n-1} + \overline{\alpha} z + 1 = 0$ have norm equal to 1.

Quickies

Answers to the Quickies are on page 397.

Q1005. *Proposed by Michael W. Botsko, Saint Vincent College, Latrobe, PA.*

Let (X, d) be a complete metric space. Suppose A and B are both G_δ sets in X and $A \cap B$ is of the first category. (A set is a G_δ set if it can be expressed as a countable intersection of open sets and it is of the first category if it can be expressed as a countable union of nowhere dense sets.) Prove that A and B cannot both be dense in X.

Q1006. *Proposed by Mowaffaq Hajja, Mathematics Department, Yarmouk University, Irbid, Jordan.*

Let ABC be a triangle and ℓ the line through A parallel to $\overline{BC}$. The internal angle bisector of $\triangle ABC$ through B intersects $\overline{AC}$ and ℓ at B' and B'', respectively. Similarly, the internal angle bisector through C intersects $\overline{AB}$ and ℓ at C' and C'', respectively. Prove that if the circumradii of $\triangle AB'B''$ and $\triangle AC'C''$ are equal, then $AB = AC$.

Solutions

When k is the largest in the cycle of 1 **December 2009**

1831. *Proposed by Emeric Deutsch, Polytechnic Institute of NYU, Brooklyn, NY.*

Let k and n be positive integers with $1 \leq k \leq n$, and let $a(n, k)$ be the number of permutations of the set $\{1, 2, \ldots, n\}$ for which k is the largest element in the cycle containing 1. Find a closed form expression for $a(n, k)$.

Solution by Jerrold W. Grossman, Department of Mathematics and Statistics, Oakland University, Rochester, MI.

The answer is $a(n, k) = (n!)/(n - k + 1)(n - k + 2)$ for $k \geq 2$ and $a(n, 1) = (n - 1)!$.

We show more generally that in a random permutation of $\{1, 2, \ldots, n\}$, the probability that l specific numbers are in the same cycle and m specific other numbers are not in that cycle is

$$\frac{m!\,(l-1)!}{(l+m)!}. \tag{1}$$

Without loss of generality we take the first subset of numbers to be $\{1, 2, \dots, l\}$ and the second subset to be $\{l + 1, l + 2, \dots, l + m\}$. We proceed by induction on n, starting with the base case $n = l + m$. Among all $(l + m)!$ permutations, there are $(l - 1)!$ ways to have one cycle consist of exactly the numbers $\{1, 2, \dots, l\}$ and $m!$ ways to have a permutation of the remaining numbers outside that cycle. Now for the inductive step we can construct a random permutation of $\{1, 2, \dots, n + 1\}$ by taking a random permutation of $\{1, 2, \dots, n\}$, written in its full cycle structure (where, by the inductive hypothesis, the probability that $1, 2, \dots, l$ are in the same cycle and $l + 1, l + 2, \dots, l + m$ are not in that cycle is given by Expression (1)), and randomly inserting the number $n + 1$ into any of the $n + 1$ positions (immediately following any of the n numbers or after the final right parenthesis). Thus the desired probability does not change and our proof is complete.

This result applies to the given problem with $l = 2$ (the numbers 1 and k) and $m = n - k$ (the numbers $k + 1$ through n). Substituting these values into Expression (1) and multiplying this probability by the number of permutations gives the answer as claimed. Note that if $k = 1$, then $l = 1$ and $m = n - 1$, giving $a(n, 1) = (n - 1)!$.

Also solved by Armstrong Problem Solvers, Michel Bataille (France), Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Australia), Robin Chapman (United Kingdom), CMC 328, Daniele Degiorgi (Switzerland), Natacha Fontes-Merz, G.R.A.20 Problem Solving Group (Italy), Omran Kouba (Syria), Jeff Lutgen, Masao Mabuchi (Japan), Matthew McMullen, Kim McInturff, José H. Nieto (Venezuela), Rob Pratt, Philip Straffin, Marian Tetiva (Romania), Dennis Walsh, Timothy Woodcock, and the proposer. There were three solutions that did not express the answer in closed form.

A quadratic cyclic system of equations **December 2009**

1832. *Proposed by Michel Bataille, Rouen, France.*

Find all solutions to the following system of equations:

$$4x^2 + 8y^2 + 2z^2 + 18xy + 8yz + 9zx = 49(x + 1)$$
$$2x^2 + 4y^2 + 8z^2 + 9xy + 18yz + 8zx = 49(y + 1)$$
$$8x^2 + 2y^2 + 4z^2 + 8xy + 9yz + 18zx = 49(z + 1)$$

Solution by Northwestern University Math Problem Solving Group, Northwestern University, Evanston, IL.

With the change of variables $x = 2u - v$, $y = 2v - w$, and $z = 2w - u$, the system becomes

$$uv - 2u + v = 1$$
$$vw - 2v + w = 1$$
$$wu - 2w + u = 1.$$

Note that if $u = -1$, then the first equation would become $2 = 1$, which is impossible, so $u \neq -1$, and similarly $v \neq -1$ and $w \neq -1$. Thus the system of equations can be rewritten as

$$v = \frac{2u + 1}{u + 1}, \quad w = \frac{2v + 1}{v + 1}, \quad \text{and} \quad u = \frac{2w + 1}{w + 1}.$$

Letting $f(t) = (2t + 1)/(t + 1)$, it follows that

$$u = f(w) = f(f(v)) = f(f(f(u))) = \frac{13u + 8}{8u + 5}.$$

Hence $u^2 - u - 1 = 0$ and thus $u = (1 \pm \sqrt{5})/2$. For each value of u the corresponding values of $v = f(u)$ and $w = f(v)$ give the two solutions to the system:

$$u = v = w = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad u = v = w = \frac{1-\sqrt{5}}{2},$$

Thus the solutions to the original system are

$$x = y = z = \frac{1+\sqrt{5}}{2} \quad \text{and} \quad x = y = z = \frac{1-\sqrt{5}}{2}.$$

Editor's Note. It is tempting to argue that just the cyclicity of the system implies that all solutions satisfy $x = y = z$. This is not true in general as shown by the system $x^2 + 2y^2 + 3z^2 = y^2 + 2z^2 + 3x^2 = z^2 + 2x^2 + 3y^2 = 24$ which has one solution $(x, y, z) = (2, 2, -2)$. Mark Ashbaugh generalized the problem by providing a method to find all solutions of any system of cyclic quadratic equations in three variables.

Also solved by George Apostolopoulos (Greece), Mark Ashbaugh, Robert Calcaterra, Robin Chapman (United Kingdom), John Christopher, CMC 328, Chip Curtis, Daniele Degiorgi (Switzerland), Michael Goldenberg and Mark Kaplan, Alex A. Griffith, Jonathan P. Hexter, Bianca-Teodora Iordache (Romania), Omran Kouba (Syria), Victor Y. Kutsenok, Kee-Wai Lau (China), Anurag Setty and Nicholas Mecholsky, Stan Wagon, Haohao Wang and Jerzy Wojdyło, and the proposer. There were three incorrect submissions.

The determinant of a Kronecker Product **December 2009**

1833. *Proposed by Sam Vandervelde and Richard Torres, St. Lawrence University, Canton, NY.*

Let p be a prime, and let M_p be the $p^2 \times p^2$ matrix whose ijth entry, $0 \le i, j \le p^2 - 1$, is given by

$$m_{i,j} = (i \bmod p)^{(j \bmod p)} \lfloor i/p \rfloor^{\lfloor j/p \rfloor},$$

where $(k \bmod p)$ denotes the remainder when k is divided by p and we take $0^0 = 1$. Prove that

$$\det(M_p) \equiv (-1)^{(p+1)/2} \pmod{p}.$$

Solution by Steven D. Smith (student) and Rick Mabry, Louisiana State University in Shreveport, Shreveport, LA.

We assume that p is odd, else the result does not hold. (It is easy to check separately that $\det(M_p) \equiv 1$ when $p = 2$.) All our variables are nonnegative integers and our equivalences are all modulo p.

The matrix M_p can be given by two identical, smaller matrices via the Kronecker product, as follows,

$$M_p = V_p \otimes V_p,$$

where V_p is the $p \times p$ matrix whose ijth entry $(0 \le i, j \le p - 1)$ is given by

$$v_{i,j} = i^j,$$

so V_p is a Vandermonde matrix. It is known that if A and B are $n \times n$ matrices, then $\det(A \otimes B) = (\det A)^n (\det B)^n$, thus

$$\det(M_p) = \det(V_p \otimes V_p) = \det(V_p)^{2p}.$$

A well-known formula for the determinant of a Vandermonde matrix gives

$$\det(V_p) = \prod_{0\le j<i<p} (i-j) = \prod_{k=1}^{p-1}(k!).$$

Furthermore,

$$\det(V_p)^2 = \prod_{k=1}^{p-1}(k!)^2 = (p-1)^2(p-2)^4(p-3)^6\cdots 2^{2(p-2)}1^{2(p-1)} \tag{1}$$

$$\equiv 1^{2p}\,2^{2p}\cdots\left(\frac{p-1}{2}\right)^{2p} = \left(\left(\frac{p-1}{2}\right)!\right)^{2p}, \tag{2}$$

where (2) is obtained by folding the sequence of terms in (1) about its center and using $(p-i)^2 \equiv i^2$, for $i = 1, 2, \ldots, (p-1)/2$. Next we 'unfold' the expression in (2); first we note that $x^{2p} \equiv x^2$ for any x, thus

$$\det(V_p)^2 \equiv \left(\left(\frac{p-1}{2}\right)!\right)^2 = \left(1\cdot 2\cdots\frac{p-1}{2}\right)\left(\frac{p-1}{2}\cdots 2\cdot 1\right),$$

next we apply the equivalence $k \equiv (-1)(p-k)$, which holds for any k, to get,

$$\det(V_p)^2 \equiv \left(1\cdot 2\cdots\frac{p-1}{2}\right)\left(\frac{p+1}{2}\cdots(p-2)(p-1)\right)(-1)^{(p-1)/2}$$
$$= (-1)^{(p-1)/2}(p-1)!.$$

Finally, by Wilson's Theorem,

$$\det(M_p) = (\det(V_p)^2)^p \equiv ((-1)^{(p-1)/2}(p-1)!)^p$$
$$\equiv ((-1)^{(p+1)/2})^p = (-1)^{(p+1)/2}.$$

Remark. The primality of p is not used through equation (1). At that point, if p is not prime we may write $p = m_1m_2$, where $1 < m_1 \le m_2 < p$. Since each of m_1, m_2 will occur as a distinct factor of $(p-1)^2(p-2)^4(p-3)^6\cdots 2^{2(p-2)}1^{2(p-1)}$ (even when $m_1 = m_2$), we see that $\det(M_p) \equiv 0$ when p is not prime.

Also solved by Michel Bataille (France), Elton Bojaxhiu (Albania) and Enkel Hysnelaj (Australia), Robert Calcaterra, Robin Chapman (United Kingdom), Omran Kouba (Syria), Elias Lampakis (Greece), Éric Pité (France), and the proposers.

The solution to Problem 1834 will appear in the next issue.

An integral identity for the Tree Function **December 2009**

1835. *Proposed by Finbarr Holland, University College Cork, Ireland.*

Let T be the so-called *tree function* defined by the power series

$$T(z) = \sum_{n=1}^{\infty} \frac{n^{n-1}}{n!} z^n.$$

For $0 \le x < \infty$ let $g(x) = T(xe^{-x})$. Show that g is continuous on $[0, \infty)$ and that if $0 < a < 1$, then

$$(1-a)\int_0^\infty \left(\frac{g(x)}{x}\right)^a dx + a\int_0^\infty \left(\frac{g(x)}{x}\right)^{1-a} dx = a(1-a)\pi^2\csc^2(\pi a).$$

Solution by Omran Kouba, Higher Institute for Applied Sciences and Technology, Damascus, Syria.

For $0 \le x < \infty$, let $\varphi(x) = xe^{-x}$. Since $\varphi'(x) = (1-x)e^{-x}$ it follows that φ is increasing on $[0, 1]$ and decreasing on $[1, \infty)$; in particular φ takes its values in $[0, 1/e]$. On the other hand, from Stirling's formula $n^{n-1}e^{-n}/n! = O(1/n^{3/2})$. It follows that the series defining T is uniformly convergent on the interval $[0, 1/e]$, and consequently T is continuous on this interval. From this we conclude that $g = T \circ \varphi$ is continuous on $[0, \infty)$.

In fact T and φ are linked in a more substantial way. Indeed, starting from φ we can define two continuous bijections:

$$\varphi_0 : [0, 1] \to [0, 1/e], x \mapsto xe^{-x} \text{ and } \varphi_1 : [1, \infty) \to (0, 1/e], x \mapsto xe^{-x},$$

and it is a well-known result [Corless et al., On the Lambert W function, *Advances in Computational Mathematics* **5** (1996) 329–359] that φ_0^{-1} is in fact the tree function T considered in this problem. Thus for $x \ge 0$, $g(x)$ is the unique solution $y \in [0, 1]$ of the equation $ye^{-y} = xe^{-x}$. In particular $g(x) = x$ for $0 \le x \le 1$.

For $x > 1$ we use the fact that $g(x)e^{-g(x)} = xe^{-x}$ to conclude that $g(x)/x = e^{-h(x)}$ where $h(x) = x - g(x)$. The function $x \mapsto h(x)$ is increasing on $[1, \infty)$ and $x - 1 \le h(x) \le x$ for $x \ge 1$. It follows that for $0 < a < 1$ and $x \ge 1$, $(g(x)/x)^a = e^{-ah(x)} \le e^a e^{-ax}$. This proves the convergence of the integral $\int_0^\infty (g(x)/x)^a\, dx$. For $0 < a < 1$ define

$$I(a) = \int_0^\infty \left(\frac{g(x)}{x}\right)^a dx = 1 + \int_1^\infty e^{-ah(x)}\, dx. \tag{1}$$

The winning idea is that h, which is a continuous increasing bijection from $[1, \infty)$ onto $[0, 1)$, has an inverse that can be easily expressed. Indeed, if $t > 0$ then there exists a unique $x > 1$ such that $h(x) = t$, and

$$\frac{x-t}{x} = \frac{x - h(x)}{x} = \frac{g(x)}{x} = e^{-h(x)} = e^{-t},$$

hence $x = h^{-1}(t) = t/(1 - e^{-t})$.

Making the change of variable $t = h(x)$ in Equation (1) and integrating by parts give

$$\begin{aligned} I(a) &= 1 + \int_0^\infty e^{-at}(h^{-1})'(t)\, dt \\ &= 1 + e^{-at}h^{-1}(t)\Big|_{t=0}^{t=\infty} + a\int_0^\infty e^{-at}h^{-1}(t)\, dt \\ &= 1 - 1 + a\int_0^\infty e^{-at}h^{-1}(t)\, dt = a\int_0^\infty \frac{te^{-at}}{1-e^{-t}}\, dt. \end{aligned}$$

Recalling that $\sum_{n=0}^{m-1} e^{-nt} = (1 - e^{-mt})/(1 - e^{-t})$ for $t > 0$ and $m > 0$, we write

$$I(a) - a\sum_{n=0}^{m-1}\int_0^\infty te^{-(a+n)t}\, dt = \int_0^\infty \frac{te^{-(a+m)t}}{1-e^{-t}} dt.$$

The simple inequality $t/(1 - e^{-t}) \le 1 + t$, which is equivalent to $1 + t \le e^t$, and the fact that $\int_0^\infty te^{-t\beta}\, dt = 1/\beta^2$ for $\beta > 0$ imply that

$$\left| I(a) - a\sum_{n=0}^{m-1} \frac{1}{(a+n)^2} \right| \le \frac{1}{a+m} + \frac{1}{(a+m)^2}.$$

Letting m tend to ∞ gives

$$I(a) = a\sum_{n=0}^{\infty}\frac{1}{(a+n)^2}.$$

Therefore, for $0 < a < 1$,

$$\begin{aligned}(1-a)I(a) + aI(1-a) &= a(1-a)\left(\sum_{n=0}^{\infty}\frac{1}{(a+n)^2} + \sum_{n=0}^{\infty}\frac{1}{(1-a+n)^2}\right)\\ &= a(1-a)\left(\frac{1}{a^2} + \sum_{n=1}^{\infty}\left(\frac{1}{(a+n)^2} + \frac{1}{(a-n)^2}\right)\right).\end{aligned}$$

Finally, the conclusion of the problem follows from the well known expansion

$$\pi^2\csc^2(\pi z) = \sum_{n\in\mathbb{Z}}\frac{1}{(z+n)^2},$$

which is valid for $z \in \mathbb{C}\setminus\mathbb{Z}$.

Editor's Note. The fact that $\varphi_0^{-1} = T$ can be easily obtained by using Lagrange's Inversion Formula.

Also solved by Paul Bracken, Robin Chapman (United Kingdom), Tony Tam, and the proposer. There was one partial solution.

Answers

Solutions to the Quickies from page 392.

A1005. Suppose both A and B are dense in X. Since A and B are both G_δ sets, there exist $\{A_n\}$ and $\{B_n\}$, two sequences of open sets, such that $A = \cap_{n=1}^{\infty}A_n$ and $B = \cap_{n=1}^{\infty}B_n$. Because A and B are dense, it follows that A_n and B_n are dense open sets for each positive integer n. Thus $C(A_n)$ and $C(B_n)$ are nowhere dense for each n. (Here $C(S)$ denotes the complement of S.) Therefore $C(A) = \cup_{n=1}^{\infty}C(A_n)$ and $C(B) = \cup_{n=1}^{\infty}C(B_n)$ are both sets of the first category, which says that $C(A)\cup C(B)$ is also of the first category. since (X, d) is a complete metric space, it is not of the first category by the Baire Category Theorem. Thus it follows that $C(C(A)\cup C(B)) = A\cap B$ is not of the first category, which is a contradiction and completes the proof.

A1006. Because $\overline{B''B}$ bisects $\angle B$ and $\overline{B''C''}$ is parallel to $\overline{BC}$, it follows that $\angle AB''B = \angle CBB'' = \angle B''BA$. Using the fact that the circumradius R of $\triangle XYZ$ is given by $R = YZ/(2\sin\angle ZXY)$, it follows that the circumradii of $\triangle AB'B''$ and $\triangle ABB'$ are equal, each respectively equal to $AB'/(2\sin\angle AB''B)$ and $AB'/(2\sin\angle B''BA)$. Similarly, the circumradii of $\triangle AC'C''$ and $\triangle AC'C$ are equal. Using the problem assumption, it follows that the circumradii of $\triangle AB'B$ and $\triangle AC'C$ are equal. Thus

$$\frac{BB'}{2\sin\angle A} = \frac{CC'}{2\sin\angle A}$$

and $BB' = CC'$. Finally, by the Steiner–Lehmus Theorem, $AB = AC$ as desired.

REVIEWS

PAUL J. CAMPBELL, *Editor*
Beloit College

Assistant Editor: Eric S. Rosenthal, West Orange, NJ. Articles, books, and other materials are selected for this section to call attention to interesting mathematical exposition that occurs outside the mainstream of mathematics literature. Readers are invited to suggest items for review to the editors.

Rehmeyer, Julie, Crowdsourcing peer review, *Science News* (9 September 2010), `http://www.sciencenews.org/index/generic/activity/view/id/63252/title/Crowdsourcing_peer_review`.

Lipton, R.J., An update on Vinay Deolalikar's "proof," `http://rjlipton.wordpress.com/2010/09/15/an-update-on-vinay-deolalikars-proof/#more-5443`.

What's the latest on $P = NP$? As you might expect, there is yet no definitive resolution of the claim in August by Vinay Deolalikar to have proved $P \neq NP$. However, great interest, coupled with the contemporary craze of Internet social networking, has produced a flood of peer-like reviews that author Rehmeyer calls "crowdsourcing." Commentators have found flaws in the proof, but Deolalikar plans to submit an amended paper to the standard review process of a journal. Journal editors may find in "crowdsourcing" a new and improved method for peer review: Instead of sending submissions to surprised referees, relying on their velleity to respond, and hammering on them after months of no response, why not just assemble a cadre of "friends" or "followers" and get instant referee reports? Indeed, author Lipton asks "how long should a claim of a great result remain unresolved?" and hopes for "some closure soon." (But what's the hurry?) Of course, not many papers in mathematics are Twitter-short; and despite the appeal to ego, how many mathematicians could be excited about refereeing yet another "Note on a Theorem in Analysis"? Moreover, thoughtful and measured reflection may be far more valuable than the result of sorting a gram of wheat from the bushel of chaff of dozens (or hundreds) of instant responses.

Reitano, Robert R., *Introduction to Quantitative Finance: A Math Tool Kit*, MIT Press, 2010; xxxiv + 709 pp, $80. ISBN 978-0-262-0369-7.

This book starts with logic, moves on through number systems to metric spaces, introduces open and closed sets, considers sequences and series and their convergence. Then it does probability (discrete and continuous) and calculus (differentiation and integration). That's a lot for one book (and particularly for the one-semester graduate course that the author recommends the book for), even though each topic is parsimoniously subsetted to the parts relevant to portfolio management and investment banking. What distinguishes the book is that results are proved; this book is not a collection of formulas: "[F]ew good careers ... depend on the application of standard formulas in standard situations. All such applications tend to be automated and run in companies' computer systems...." What makes it valuable for mathematicians striving for relevance is that each chapter concludes with a section on applications to finance (sometimes longer than the rest of the chapter). There are exercises, and both a student solutions manual and an instructor's guide are available. This could be a useful book for a topics course in quantitative finance for senior mathematics majors (non-majors wouldn't be able to handle it). (Pace Euclid, is greed the best motivation for mathematics?)

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Mahajan, Sanjoy, *Street-Fighting Mathematics: The Art of Educated Guessing and Opportunistic Problem Solving*, MIT Press, 2010; xv + 134 pp, $25 (P). ISBN 0–978–0–26251429–3. Free download under Creative Commons license at `http://mitpress.mit.edu/books/full_pdfs/Street-Fighting_Mathematics.pdf`.

Bartlett, Tom, The gospel of well-educated guessing, *Chronicle of Higher Education* (2 May 2010), `http://chronicle.com/article/The-Gospel-of-Well-Educated/65351/`.

Dubner, Stephen J., Surely you must be joking, Mr. Mahajan! The "street-fighting mathematician" answers your questions. Freakonomics (*New York Times* blog) (16 July 2010), `http://freakonomics.blogs.nytimes.com/2010/07/16/surely-you-must-be-joking-mr-mahajan-the-street-fighting-mathematician-answers-your-questions/`.

Don't be put off by this book's Foreword (it's not by author Mahajan), which begins ominously: "Most of us took mathematics courses from mathematicians—Bad Idea! Mathematicians see mathematics as an area of study in its own right. . . . [M]athematics courses, as they are taught today, are seldom helpful and are often downright destructive." Well, those certainly are fighting words! Fortunately, although in support of the title the author touts "shoot first and ask questions later," that advice is in the spirit of a "valuable [mathematical] problem-solving philosophy." The book's subtitle is accurate—the book is about educated guessing in mathematics. It begins with dimensional analysis (doing without is "like fighting with one hand tied behind our back"); proceeds to "easy cases" (extreme cases for integrals, areas, volumes, and drag); uses "lumping" for estimation (from which Stirling's approximation results); recommends "seeing an idea" via pictorial proofs (the AGM inequality emerges); uses "low-entropy" expressions for successive approximations (try $\int_{-\pi/2}^{\pi/2}(\cos t)^{100}\,dt$); and applies Polya's advice to solve a simpler problem first (Euler-Maclaurin summation makes an appearance). From the contents, you can see that this book is for the aspiring streetwise mathematician—but one who must already have mastered the jujitsu of calculus. (Publishers: Although Mahajan notes in the long Freakonomics interview that his next book will be *Street-Fighting Tools for Science and Engineering*, please, no more provocative titles! We can do without *Back-Alley Boolean Algebra*, *Kung Fu Functors*, *No-Holes-Barred Homology*, *Mixed Mathematical-Martial Arts*, *Jedi Set Theory*, *Weierstrass Wrestlingmania*, etc., ad nauseam—or even ad humorem.)

Elwes, Richard, To infinity and beyond: The struggle to save arithmetic, *New Scientist* (2773) (16 August 2010).

Valgreen, Jesper, Letters: Straight sets, *New Scientist* (2781) (6 October 2010).

Stillwell, John, *Roads to Infinity: The Mathematics of Truth and Proof*, A K Peters, 2010; xi + 203 pp, $39. ISBN 978-1-56881-466-7.

Gödel's incompleteness theorem says that some true statements in arithmetic cannot be proved from the Peano axioms, and Gödel gave a self-referential example. Are there more-prosaic examples of "unprovable theorems"? Jeff Paris and Leo Harrington in 1977 gave one based on a variation of Ramsey's theorem, and others can be derived from Kruskal's theorem about tree embeddings and the related graph minor problem. Now Harvey Friedman (Ohio State Univ.) has found other examples, based on *expansive linear growth* (ELG) functions, a subclass of *strictly dominating* functions (ones for which the output is larger than the input). However, even the statements about ELG functions can be proved if one assumes the existence of large cardinal numbers; so Friedman asserts that large cardinals will be a conventional and essential part of "concrete mathematics" in the future. Is the choice then between belief in large cardinals vs. unease with undecidable statements? Is it Scylla vs. Charybdis? The Pit of infinity, or the Pendulum that cuts us off from truths? The response letter from Valgreen contends that a "finitist" mathematics does not lead to just "finite mathematics" but indeed still allows consideration of potentially infinite sets, such as the natural numbers (which even Kronecker didn't want to lose). Because Friedman's own work is both long and esoteric, Elwes's interpretive article is valuable, as is Stillwell's book, which recounts—in remarkably understandable fashion!—the history of infinity and provability to just months before Friedman's latest results.

NEWS AND LETTERS

Acknowledgments

In addition to our Associate Editors, the following referees have assisted the MAGAZINE *during the past year. We thank them for their time and care.*

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